The University of Chicago

STABILITY OF PERMANENT CONFIGURATIONS IN THE PROBLEM OF FOUR BODIES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1939

By
ADRIENNE SOPHIE RAYL

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS
1942

The University of Chicago

STABILITY OF PERMANENT CONFIGURATIONS IN THE PROBLEM OF FOUR BODIES

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1939

By
ADRIENNE SOPHIE RAYL

Private Edition, Distributed by
THE UNIVERSITY OF CHICAGO LIBRARIES
CHICAGO, ILLINOIS
1942

Digitized by the Internet Archive
in 2026

https://archive.org/details/IA41544324_0087

TABLE OF CONTENTS

Section Page

INTRODUCTION 1

CHAPTER I
STABILITY OF THE ISOSCELES TRAPEZOID

1. The differential equations of motion 3

2. Properties 3

3. The equations of variation 6

4. The characteristic equations 9

5. Existence of roots 0, 0, $\pm i\omega$ 10

6. The expansion of the characteristic equation of the z-system 11

7. The characteristic equation of the x,y-system for the square 12

8. The characteristic equation of the x,y-system for the equilateral triangle 14

9. The characteristic equation of the x,y-system for the general isosceles trapezoid 18

10. The characteristic exponents for the x,y-system . . 28

11. Periodic solutions 32

CHAPTER II
STABILITY OF THE EQUILATERAL TRIANGLE WITH CENTRAL BODY

12. Properties 38

13. The equations of variation 40

14. The characteristic equation of the z-system 41

Section Page

15. The characteristic equation of the x,y-system . . . 41

16. Limiting case of three bodies 43

17. The characteristic exponents for the x,y-system . . 44

18. Periodic solutions 51

SUMMARY 55

INTRODUCTION

An analysis of the quadrilateral configurations of four bodies was published a few years ago by W. D. MacMillan and W. Bartky.[1] Their paper contains an investigation into the properties of quadrilaterals which, for properly chosen masses, form permanent configurations. Such quadrilaterals, which are termed admissible, include certain convex and certain concave configurations. Our attention in this paper will be centered upon two of the particular admissible quadrilaterals which have been found by MacMillan and Bartky to exist provided the masses fulfill certain conditions. The first one to be considered, the isosceles trapezoid with the four bodies lying at the vertices, constitutes an admissible convex configuration. The second one in which we shall be interested, the equilateral triangle with three bodies at the vertices and the fourth at the center, is an admissible concave configuration. Our discussion will thus be limited to configurations possessing rather simple geometric properties. An inquiry into the stability of the other admissible quadrilaterals would evidently be much more complex in character.

It is the purpose of the first part of this paper to investigate for the isosceles trapezoid solution the existence of periodic orbits of the four bodies about their points at the vertices

---

[1] W. D. MacMillan and Walter Bartky, Permanent Configurations in the Problem of Four Bodies, Transactions of the American Mathematical Society, vol. 34 (1932), pp. 838-875.

of the isosceles trapezoid. In the second part, we shall inquire into the existence of periodic orbits for the case of the equilateral triangle with central body. The permanent configuration, in either case, gives a particular solution of the differential equations of motion of four bodies. We shall derive the equations of variation for these particular solutions of the equations of motion and then follow the usual method of determining the existence of periodic orbits about the permanent configuration points through an investigation into the periodic solutions of the equations of variation. The problem as here outlined is equivalent to investigating the stability of the configurations.

Following the announcement by Lagrange of the straight line and equilateral triangle configurations, the determination of periodic orbits in the problem of three bodies became a field for investigation. A detailed treatment of periodic orbits for three bodies when one of the masses is infinitesmal has been given by F. R. Moulton.[1] Results for three finite bodies have been published by H. E. Buchanan.[2]

---

[1] F. R. Moulton, Periodic Orbits, (Washington: Carnegie Institution of Washington, 1920).

[2] H. E. Buchanan, Periodic Oscillations of Three Finite Masses about the Lagrangian Circular Solutions, American Journal of Mathematics, vol. 45 (1923), pp. 93-121.

H. E. Buchanan, Periodic Orbits of Three Finite Masses about the Equilateral Triangle Points, American Journal of Mathematics, vol. 50 (1928), pp. 613-626.

CHAPTER I

STABILITY OF THE ISOSCELES TRAPEZOID

1. The differential equations of motion. Let m_1, m_2, m_3, m_4 represent the masses. Let ξ_1, η_1, ζ_1 $(i = 1, 2, 3, 4)$ be the coordinates of the i-the body with the center of gravity of the four masses as the origin and with axes rotating uniformly in the ξ,η -plane with angular velocity ω. Let the distance between m_i and m_j be given by r_{ij} $(i, j = 1, 2, 3, 4;\ i \neq j)$. Then the differential equations of motion which follow from Newton's law of gravitation are

$$
\begin{aligned}
&\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} = \omega^2\xi_i + \frac{1}{m_i}\frac{\partial U}{\partial \xi_i}, \\
&\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} = \omega^2\eta_i + \frac{1}{m_i}\frac{\partial U}{\partial \eta_i}, \qquad (1) \\
&\frac{d^2\zeta_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \zeta_i} \qquad (i = 1, 2, 3, 4),
\end{aligned}
$$

where

$$
U = \frac{1}{2}\sum_{i,j=1}^{4}\frac{m_i m_j}{r_{ij}} \qquad (i \neq j).
$$

2. Properties. We assume that the bodies lie in the order m_1, m_2, m_3, m_4 in a counter-clockwise direction around the quadrilateral which they determine. We assume also that

$$
m_1 = m_2, \qquad m_3 = m_4. \qquad (2)
$$

Then there exists one and only one isosceles trapezoid configuration.[1]

---

[1] MacMillan and Bartky, op.cit., p. 866.

We adopt the notation $r_1 = r_{12}$, $r_2 = r_{23}$, $r_3 = r_{34}$, $r_4 = r_{41}$, $r_5 = r_{13}$, $r_6 = r_{24}$. Let the distance r_o be defined by

$$\omega^2 = \frac{M}{r_o^3},$$

where

$$M = 2m_1 + 2m_3.$$

Let

$$R_i = \frac{1}{r_i^3} \qquad (i = 0, 1, \cdots, 6). \tag{3}$$

We may then write

$$\omega^2 = MR_o. \tag{4}$$

The following relations[1] exist among the R_i:

$$R_2 + R_5 = 2R_o, \tag{5}$$

$$(R_1 - R_o)(R_3 - R_o) = (R_2 - R_o)^2 = (R_o - R_5)^2. \tag{6}$$

The ratio of the masses is given by[2]

$$\frac{m_3}{m_1} = \frac{r_1}{r_3} \cdot \frac{R_1 - R_o}{R_2 - R_o}, \tag{7}$$

or

$$\frac{m_3}{m_1} = \left(\frac{r_3}{r_1} \cdot \frac{r_o^3 - r_1^3}{r_o^3 - r_3^3} \right)^{1/2}. \tag{8}$$

The coordinates of the bodies are given by the constants

$$\xi_i^{(o)}, \qquad \eta_i^{(o)}, \qquad \zeta_i^{(o)},$$

where

$$\xi_1^{(o)} = -\frac{r_1}{2}, \quad \xi_2^{(o)} = \frac{r_1}{2}, \quad \xi_3^{(o)} = \frac{r_3}{2}, \quad \xi_4^{(o)} = -\frac{r_3}{2},$$

$$\eta_1^{(o)} = \eta_2^{(o)}, \quad \eta_3^{(o)} = \eta_4^{(o)}, \quad \zeta_i^{(o)} = 0 \qquad (i = 1, 2, 3, 4).$$

---

[1] Ibid., p. 863.

[2] Ibid., p. 866.

The proof that

$$\xi_1 = \xi_1^{(o)}, \quad \eta_1 = \eta_1^{(o)}, \quad \dot{\xi}_1 = \dot{\xi}_1^{(o)}$$

satisfy the differential equations (1) is made with the aid of (5), (6), and (7). It is to be noted that the isosceles trapezoid lies in the ξ, η -plane and that r_1 and r_3 are parallel to each other and to the ξ -axis. Then $r_2 = r_4$ and $r_5 = r_6$. It follows immediately from these equalities that

$$R_2 = R_4, \qquad R_5 = R_6. \tag{9}$$

MacMillan and Bartky defined K by the relation[1]

$$r_2 = K(r_1 r_3)^{1/2}, \tag{10}$$

and showed that as K increases from 1 to ∞ the trapezoid changes continuously from a square to an equilateral triangle[2] and the ratio m_3/m_1 decreases[3] from 1 to 0.

Some other relations taken from the paper by MacMillan and Bartky for which we shall have need are[4]

$$\frac{R_o}{(R_1 R_3)^{1/2}} = \frac{1}{2}\left\{K^{-3} + (1 + K^2)^{-3/2}\right\}, \tag{11}$$

$$r_1^3 = \frac{1}{2}\left[1 + ab \pm \{(1 - a^2)(1 - b^2)\}^{1/2}\right] r_o^3, \tag{12}$$

$$r_3^3 = \frac{1}{2}\left[1 + ab \mp \{(1 - a^2)(1 - b^2)\}^{1/2}\right] r_o^3, \tag{13}$$

$$r_2 = \left(\frac{a + b}{2a}\right)^{1/3} r_o, \tag{14}$$

---

[1] Ibid., p. 863.

[2] Ibid., p. 865.

[3] Ibid., p. 866.

[4] Ibid., p. 863-864.

$$r_5 = \left(\frac{a+b}{2b}\right)^{1/3} r_0, \tag{15}$$

where

$$a = K^{-3}, \qquad b = (1+K^2)^{-3/2}. \tag{16}$$

As K increases from 1 to ∞, it is seen from (16) that a decreases from 1 to 0. Therefore as a varies from 1 to 0, the trapezoid changes continuously from a square to an equilateral triangle.

There is no loss in generality in assuming

$$r_1 \geqq r_3.$$

We shall therefore use the upper signs in equations (12) and (13). The unit of distance may be so chosen that $r_1 = 1$, and the unit of mass so that $m_1 = 1$. By virtue of the choice of the unit of distance and through the use of (3), the relations (11) through (15) may be put into the forms

$$\frac{R_0}{R_3^{1/2}} = \frac{a+b}{2}, \tag{17}$$

$$R_0 = \frac{1+ab+\sqrt{(1-a^2)(1-b^2)}}{2}, \tag{18}$$

$$R_3 = \frac{1+ab+\sqrt{(1-a^2)(1-b^2)}}{1+ab-\sqrt{(1-a^2)(1-b^2)}}, \tag{19}$$

$$R_2 = R_0 \cdot \frac{2a}{a+b}, \tag{20}$$

$$R_5 = R_0 \cdot \frac{2b}{a+b}. \tag{21}$$

3. The equations of variation. If m_2, m_4, R_4, and R_6 are eliminated by the use of (2) and (9) and the variables ξ_4, η_4, ζ_4 are eliminated by means of the center of gravity integrals, the equations of variation for the isosceles trapezoid solution are

$$(22)\qquad \begin{cases} \dfrac{d^2x_i}{dt^2} - 2\omega \dfrac{dy_i}{dt} = A_{ij}x_j + B_{ij}y_j, \\ \dfrac{d^2y_i}{dt^2} + 2\omega \dfrac{dx_i}{dt} = C_{ij}x_j + D_{ij}y_j, \end{cases}$$

$$(23)\qquad \frac{d^2z_i}{dt^2} = E_{ij}z_j \qquad (i,j = 1, 2, 3),$$

where the repeated subscripts indicate summation and where

$$A_{11} = \omega^2 + 2m_1R_1 + m_3(2R_5 - T_5) + M(R_2 - T_2/2),$$

$$A_{12} = m_1(-2R_1 + 2R_2 - T_2),$$

$$A_{13} = m_3(2R_2 - 2R_5 - T_2 + T_5),$$

$$A_{21} = m_1(-2R_1 + 2R_5 - T_5),$$

$$A_{22} = \omega^2 + 2m_1R_1 + m_3(2R_2 - T_2) + M(R_5 - T_5/2),$$

$$A_{23} = m_3(-2R_2 + 2R_5 + T_2 - T_5),$$

$$A_{31} = m_1(2R_3 - 2R_5 + T_5),$$

$$A_{32} = m_1(-2R_2 + 2R_3 + T_2),$$

$$A_{33} = \omega^2 + m_1(2R_2 + 2R_5 - T_2 - T_5) + 4m_3R_3;$$

$$(24)\qquad \begin{aligned} B_{11} &= M/2\ V_2 + m_3V_5, \\ B_{12} &= B_{32} = m_1V_2, \\ B_{13} &= B_{23} = m_3(V_2 - V_5), \\ B_{31} &= B_{21} = -m_1V_5, \\ B_{22} &= -m_3V_2 - M/2\ V_5, \end{aligned}$$

$$B_{33} = m_1(-V_2 + V_5);$$

$$C_{ij} = B_{ij} \qquad (i,j = 1, 2, 3);$$

$$D_{11} = \omega^2 - m_1R_1 + m_3(-R_5 + T_5) + M/2\ (-R_2 + T_2),$$

$$D_{12} = m_1(R_1 - R_2 + T_2),$$

$$D_{13} = m_3(-R_2 + R_5 + T_2 - T_5),$$

$$D_{21} = m_1(R_1 - R_5 + T_5),$$

$$D_{22} = \omega^2 - m_1R_1 + m_3(-R_2 + T_2) + M/2\ (-R_5 + T_5),$$

$$D_{23} = m_3(R_2 - R_5 - T_2 + T_5),$$

$$D_{31} = m_1(-R_3 + R_5 - T_5),$$

$$D_{32} = m_1(R_2 - R_3 - T_2),$$

$$D_{33} = \omega^2 + m_1(-R_2 - R_5 + T_2 + T_5) - 2m_3R_3;$$

$$E_{11} = -m_1R_1 - m_3R_5 - M/2\ R_2,$$

$$E_{12} = m_1R_1 - m_1R_2,$$

$$E_{13} = m_3R_5 - m_3R_2,$$

$$E_{21} = m_1R_1 - m_1R_5,$$

(24') $$E_{22} = -m_1R_1 - m_3R_2 - M/2\ R_5,$$

$$E_{23} = m_3R_2 - m_3R_5,$$

$$E_{31} = m_1R_5 - m_1R_3,$$

$$E_{32} = m_1R_2 - m_1R_3 ,$$

$$E_{33} = -m_1R_5 - m_1R_2 - 2m_3R_3 .$$

The notations T_2, T_5, V_2, V_5 are defined by

$$T_2 = \frac{3r_7^2}{r_2^5} , \qquad T_5 = \frac{3r_7^2}{r_5^5} ,$$

$$V_2 = \frac{3r_7(r_1 - r_3)}{2r_2^5} , \qquad V_5 = \frac{3r_7(r_1 + r_3)}{2r_5^5} , \tag{25}$$

where r_7 is the perpendicular distance between r_1 and r_3.

4. The characteristic equations. The characteristic equation for the system (22) is

$$\begin{vmatrix} -\lambda^2 I + A & 2\omega\lambda I + B \\ -2\omega\lambda I + C & -\lambda^2 I + D \end{vmatrix} = 0 , \tag{26}$$

where I is the 3-rowed identity matrix and

$$A = (A_{ij}), \quad B = (B_{ij}), \quad C = (C_{ij}), \quad D = (D_{ij}) \quad (i,j = 1, 2, 3).$$

In computing the characteristic equation (26) we shall precede the computation for the general isosceles trapezoid by the expansion for the two limiting positions; namely, for the square and for the equilateral triangle. These two comparatively simple forms will then serve as a check and as an aid in the analysis of the more complicated general form.

The characteristic equation for the system (23) is

$$| -\lambda^2 I + E | = 0, \tag{27}$$

where

$$E = (E_{ij}) \qquad (i,j = 1, 2, 3).$$

5. Existence of roots 0, 0, $\pm i\omega$. H. E. Buchanan and W. L. Duren, Jr. found particular values of the characteristic exponents which must appear when given integrals of the equations of variation possess certain properties.[1] We shall apply their results to our present problem.

Let us introduce new variables into the system (1) by the transformation

$$\xi_1 = \xi_1^{(0)} + \bar{x}_1, \qquad \eta_1 = \eta_1^{(0)} + \bar{y}_1, \qquad \zeta_1 = \bar{z}_1.$$

The three integrals of areas and the energy integral are then given by

$$F_1(\bar{x},\bar{y},\bar{z},t) = \sum_{i=1}^{4} m_i[\omega\{(\xi_i^{(0)} + \bar{x}_1)^2 + (\eta_1^{(0)} + \bar{y}_1)^2\} + (\xi_1^{(0)} + \bar{x}_1)\, d\bar{y}_1/dt - (\eta_1^{(0)} + \bar{y}_1)\, d\bar{x}_1/dt],$$

$$F_2(\bar{x},\bar{y},\bar{z},t) = \sum_{i=1}^{4} m_i\,[(\xi_1^{(0)} + \bar{x}_1)\, d\bar{z}_1/dt - \bar{z}_1\, d\bar{x}_1/dt + \omega(\eta_1^{(0)} + \bar{y}_1)\bar{z}_1]\sin\omega t + \sum_{i=1}^{4} m_i[(\eta_1^{(0)} + \bar{y}_1)\, d\bar{z}_1/dt - \bar{z}_1\, d\bar{y}_1/dt - \omega(\xi_1^{(0)} + \bar{x}_1)\bar{z}_1]\cos\omega t,$$

$$F_3(\bar{x},\bar{y},\bar{z},t) = \sum_{i=1}^{4} m_i[-(\eta_1^{(0)} + \bar{y}_1)\, d\bar{z}_1/dt + \bar{z}_1\, d\bar{y}_1/dt + \omega(\xi_1^{(0)} + \bar{x}_1)\bar{z}_1]\sin\omega t + \sum_{i=1}^{4} m_i[(\xi_1^{(0)} + \bar{x}_1)\, d\bar{z}_1/dt - \bar{z}_1\, d\bar{x}_1/dt + \omega(\eta_1^{(0)} + \bar{y}_1)\bar{z}_1]\cos\omega t,$$

$$F_4(\bar{x},\bar{y},\bar{z},t) = 1/2 \sum_{i=1}^{4} m_i[(d\bar{x}_1/dt)^2 + (d\bar{y}_1/dt)^2 + (d\bar{z}_1/dt)^2 - \omega^2\{(\xi_1^{(0)} + \bar{x}_1)^2 + (\eta_1^{(0)} + \bar{y}_1)^2\}] - U.$$

---

[1] H. E. Buchanan and W. L. Duren, Jr., On the Characteristic Exponents in Certain Types of Problems of Mechanics, Duke Mathematical Journal, vol. 1 (1935), pp. 436-441.

The variables $\bar{x}_4$, $\bar{y}_4$, $\bar{z}_4$ may be eliminated by means of the center of gravity integrals. The equations of variation then have the following integrals[1]

$$(28)\qquad \sum_{i=1}^{3}\left[\frac{\partial}{\partial \bar{x}_i}F_k(0,0,0,t)x_1 + \frac{\partial}{\partial \bar{y}_i}F_k(0,0,0,t)y_1 + \frac{\partial}{\partial \bar{z}_i}F_k(0,0,0,t)z_i)\right] \qquad (k = 1,2,3,4).$$

Since two of the integrals (28) are independent of the time, two of the characteristic exponents for the equations of variation are zero.[2] As the other two integrals are periodic with minimum period $2\pi/\omega$, two other characteristic exponents[3] are $\omega\sqrt{-1}$ and $-\omega\sqrt{-1}$.

6. The expansion of the characteristic equation of the z-system. Equation (27) may be put into the form

$$\begin{vmatrix} -\lambda^2 - \frac{M}{2}R_2 - \frac{M}{2}R_5 & 0 & 0 \\ m_1R_1 - m_1R_5 & -\lambda^2 - 2m_1R_1 - m_3R_2 - m_3R_5 & m_3R_2 - m_3R_5 \\ -m_1R_3 + m_1R_5 & m_1R_2 - m_1R_5 & -\lambda^2 - m_1R_2 - m_1R_5 - 2m_3R_3 \end{vmatrix} = 0.$$

Through the use of (4), (5), and (6), this becomes

$$(\lambda^2 + \omega^2)[\lambda^4 + \lambda^2(\omega^2 + 2m_1R_1 + 2m_3R_3) + \omega^2(2m_1R_1 + 2m_3R_3)] = 0,$$

or

$$(\lambda^2 + \omega^2)^2(\lambda^2 + 2m_1R_1 + 2m_3R_3) = 0.$$

---

[1] Ibid., p. 439

[2] Ibid., p. 439.

[3] Ibid., p. 440.

Since we have assumed $r_3 \leqq r_1$ and since the relation[1] $r_1 \leqq r_o$ exists, it follows that

$$MR_o \leqq MR_1 \leqq 2m_1R_1 + 2m_3R_3.$$

The two equalities never occur simultaneously as the inequality $R_o < R_1$ holds for the square and $R_1 < R_3$ for all other cases. Therefore ω^2 and $2m_1R_1 + 2m_3R_3$ are always distinct.

The roots of (27) are

$$\lambda = \pm i\omega, \pm i\omega, \pm i\sqrt{2m_1R_1 + 2m_3R_3}, \tag{29}$$

which are all pure imaginary.

It is to be observed that m_3R_3 is an indeterminate form for the triangle since m_3 is zero and R_3 is infinite. However it is proven later that it may be evaluated with the result that m_3R_3 is equal to 3/4 for the triangle.

7. The characteristic equation of the x,y-system for the square. The quantities which occur in the coefficients of the characteristic equations have the following values for the square

$$\begin{aligned} &R_1 = R_2 = R_3 = 1, && R_5 = 1/2\sqrt{2}, \\ &\omega^2 = 2 + 1/\sqrt{2}, \\ &m_1 = m_3 = 1, && M = 4, \\ &T_2 = 3, \quad V_2 = 0, && T_5 = V_5 = 3/4\sqrt{2}. \end{aligned} \tag{30}$$

Equation (26) for the square is

$$\begin{vmatrix} -\lambda^2 I + A' & 2\omega\lambda I + B' \\ -2\omega\lambda I + B' & -\lambda^2 I + D' \end{vmatrix} = 0,$$

---

[1] MacMillan and Bartky, op.cit., p. 853.

where

$$A' = \begin{pmatrix} \omega^2 + 1/4\sqrt{2} & -3 & -1 - 1/4\sqrt{2} \\ -2 + 1/4\sqrt{2} & 3/2\ \omega^2 & 1 + 1/4\sqrt{2} \\ 2 - 1/4\sqrt{2} & 3 & 5/4\ \omega^2 + 5/2 \end{pmatrix},$$

$$B' = \begin{pmatrix} 3/4\sqrt{2} & 0 & -3/4\sqrt{2} \\ -3/4\sqrt{2} & -3/2\sqrt{2} & -3/4\sqrt{2} \\ -3/4\sqrt{2} & 0 & 3/4\sqrt{2} \end{pmatrix},$$

$$D' = \begin{pmatrix} 5/4\ \omega^2 + 5/2 & 3 & 2 - 1/4\sqrt{2} \\ 1 + 1/4\sqrt{2} & 3/2\ \omega^2 & -2 + 1/4\sqrt{2} \\ -1 - 1/4\sqrt{2} & -3 & \omega^2 + 1/4\sqrt{2} \end{pmatrix},$$

This may be put into the form

$$[(-\lambda^2 + \omega^2 + 2)^2 + 4\omega^2\lambda^2]\ N = 0,$$

where

$$N = \begin{vmatrix} 0 & -\lambda^2 + \frac{3}{2\sqrt{2}} & 2\omega\lambda - \frac{3}{2\sqrt{2}} & -4\omega\lambda \\ \lambda^2 - \frac{3}{2}\omega^2 - 3 & 3 & 0 & 2\omega\lambda + \frac{3}{2\sqrt{2}} \\ -2\omega\lambda + \frac{3}{2\sqrt{2}} & 0 & 3 & \lambda^2 - \frac{3}{2}\omega^2 - 3 \\ 4\omega\lambda & -2\omega\lambda - \frac{3}{2\sqrt{2}} & -\lambda^2 + \frac{3}{2\sqrt{2}} & 0 \end{vmatrix}.$$

We then have

$$(31)\quad (\lambda^4 + \sqrt{2}\lambda^2 + 33/2 + 4\sqrt{2})(\lambda^4 + \omega^2\lambda^2)(\lambda^4 + \omega^2\lambda^2 + 9\sqrt{2}) = 0,$$

or

$$(32)\quad \begin{aligned}\lambda^{12} &+ (4 + 2\sqrt{2})\lambda^{10} + (23 + 19\sqrt{2})\lambda^8 + (105 + 55\sqrt{2})\lambda^6 \\ &+ (198\ 1/4 + 417\sqrt{2}/2)\lambda^4 + (292\ 1/2 + 333\sqrt{2})\lambda^2 = 0.\end{aligned}$$

The roots of (26) for the square are therefore

$$(33)\quad \lambda = 0,\ 0,\ \pm i\omega,\ \pm\sqrt{\frac{-\omega^2 \pm \sqrt{\omega^4 - 36\sqrt{2}}}{2}},\ \pm\sqrt{\frac{-\sqrt{2} \pm 4\sqrt{-4 - \sqrt{2}}}{2}}.$$

8. The characteristic equation of the x,y-system for the equilateral triangle. In order to obtain the characteristic equations for the equilateral triangle, the indeterminate form m_3R_3 must be evaluated. Since $m_1 = 1$ and $r_1 = 1$, we may write (8) in the form

$$m_3 = \left(r_3 \cdot \frac{r_0^3 - 1}{r_0^3 - r_3^3} \right)^{1/2}.$$

By virtue of (3) and (17), we have

$$m_3R_3 = F\left[\frac{1 - R_0}{(a + b)^{10/3}}\right]^{1/2},$$

where

$$F = \frac{2^{8/3}R_0^{13/6}}{[4R_0 - (a + b)^2]^{1/2}}.$$

Through the use of (18) and then (15), we may write

$$m_3R_3 = F\left[\frac{1 - ab - \sqrt{(1 - a^2)(1 - b^2)}}{2(a + b)^{10/3}}\right]^{1/2} = FG\,\frac{a - b}{(a + b)^{5/3}}$$

$$= FG\left(\frac{b}{a + b}\right)^{5/3} \cdot \frac{a - b}{b^{5/3}} = FG\left(\frac{r_o}{2^{1/3}r_5}\right)^5 \cdot \frac{a - b}{b^{5/3}},$$

where

$$G = \frac{1}{2^{1/2}(1 - ab + \sqrt{(1 - a^2)(1 - b^2)}\,)^{1/2}}.$$

We replace a and b in

$$\frac{a - b}{b^{5/3}}$$

by their values in terms of K and obtain

$$\frac{a - b}{b^{5/3}} = \frac{K^{-3} - (1 + K^2)^{-3/2}}{[(1 + K^2)^{-3/2}]5/3} = \frac{K^2[1 - (1 + 1/K^2)^{-3/2}]}{(1 + 1/K^2)^{-5/2}}.$$

Since the binomial expansion of $(1 + 1/K^2)^m$ converges for $K > 1$, this may be put into the form

$$\frac{a - b}{b^{5/3}} = \frac{K^2[1 - (1 - 3/2K^2 + 15/8K^4 \cdots)]}{1 - 5/2k^2 + 35/8K^4 \cdots}$$

$$= \frac{3/2 - 15/8K^2 \cdots}{1 - 5/2K^2 \cdots}, \qquad K > 1.$$

We have finally

$$\lim_{K\to\infty} m_3R_3 = \lim_{K\to\infty} FG\left(\frac{r_o}{2^{1/3}r_5}\right)^5 \left(\frac{3/2 - 15/8K^2 \cdots}{1 - 5/2K^2 \cdots}\right)$$

$$= (2^{5/3})(1/2)(1/2)^{5/3}(3/2) = 3/4.$$

The quantities which occur in the coefficients of the characteristic equations have the following values for the equilateral triangle,

$$
\begin{aligned}
&R_1 = R_2 = R_5 = 1, && R_3 = \infty,\\
&m_3 = 0, \qquad m_1 = 1, && M = 2,\\
&\omega^2 = 2, && m_3R_3 = 3/4,\\
&T_2 = T_5 = 9/4, && V_2 = V_5 = 3\sqrt{3}/4.
\end{aligned}
\tag{34}
$$

We replace R_1, R_2, R_5, m_1, M, T_2, T_5, V_2, V_5 by their values as here given wherever they occur in the characteristic equation (26) for the equilateral triangle. This equation may then be put into the form

$$
\begin{vmatrix} -\lambda^2 I + A' & 2\omega\lambda I + B' \\ -2\omega\lambda I + B' & -\lambda^2 I + D' \end{vmatrix} = 0,
$$

where

$$
A' = \begin{pmatrix} \omega^2 + 4 - m_3/2 & -9/4 & 0 \\ 0 & \omega^2 - 1/2 - m_3/2 & 0 \\ 0 & 1/4 + 2R_3 & \omega^2 + 4m_3R_3 - 1/2 \end{pmatrix},
$$

$$
B' = \begin{pmatrix} 3\sqrt{3}\, m_3/2 & 3\sqrt{3}/4 & 0 \\ 3\sqrt{3}\, m_3 & -3\sqrt{3}\, m_3/2 & 0 \\ -3\sqrt{3}/2 & 3\sqrt{3}/4 & 0 \end{pmatrix},
$$

$$
D' = \begin{pmatrix} \omega^2 + 5/2\, m_3 - 2 & 9/4 & 0 \\ 0 & \omega^2 + 5/2\, m_3 + 5/2 & 0 \\ 0 & -5/4 - R_3 & \omega^2 + 5/2 - 2m_3R_3 \end{pmatrix}.
$$

We then obtain

$$[(-\lambda^2+\omega^2+4m_3R_3-1/2)(-\lambda^2+\omega^2+5/2-2m_3R_3)+4\omega^2\lambda^2]N=0,$$

where

$$N=\begin{vmatrix} -\lambda^2+\omega^2+4-\frac{m_3}{2} & -\frac{9}{4} & 2\omega\lambda+\frac{3\sqrt{3}m_3}{2} & \frac{3\sqrt{3}}{4} \\ 0 & -\lambda^2+\omega^2-\frac{1}{2}-\frac{m_3}{2} & 3\sqrt{3}\,m_3 & 2\omega\lambda-\frac{3\sqrt{3}m_3}{2} \\ -2\omega\lambda+\frac{3\sqrt{3}m_3}{2} & \frac{3\sqrt{3}}{4} & -\lambda^2+\omega^2+\frac{5m_3}{2}-2 & \frac{9}{4} \\ 3\sqrt{3}\,m_3 & -2\omega\lambda-\frac{3\sqrt{3}m_3}{2} & 0 & -\lambda^2+\omega^2+\frac{5m_3}{2}+\frac{5}{2} \end{vmatrix}.$$

Replacing m_3R_3 by 3/4, and m_3 by 0, and completing the expansion, we have

$$(35)\qquad (\lambda^4+\lambda^2/2+27/2)(\lambda^4+\omega^2\lambda^2)(\lambda^4+\omega^2\lambda^2+27/4)=0,$$

or

$$(36)\qquad \lambda^{12}+\frac{9}{2}\lambda^{10}+26\tfrac{1}{4}\lambda^8+72\tfrac{7}{8}\lambda^6+151\tfrac{7}{8}\lambda^4+182\tfrac{1}{4}\lambda^2=0.$$

The roots of (26) for the equilateral triangle are therefore

$$(37)\qquad \lambda=0,\ 0,\ \pm i\omega,\ \pm\sqrt{\frac{-2\pm i\sqrt{23}}{2}},\ \pm\sqrt{\frac{-1\pm i\sqrt{215}}{4}}.$$

A comparison of these results with known results in the problem of three bodies is of interest at this point. The following equation was obtained by Buchanan[1] for the equilateral triangle configuration involving three finite bodies

---

[1] Buchanan, Periodic Orbits of Three Finite Masses about the Equilateral Triangle Points, op.cit., p. 617.

$$(\lambda^4 + \omega^2\lambda^2)[\lambda^4 + \omega^2\lambda^2 + 27/4\ (m_1m_2 + m_1m_3 + m_2m_3)] = 0.$$

When we put $m_1 = m_2 = 1$, $m_3 = 0$, this becomes

$$(38) \qquad (\lambda^4 + \omega^2\lambda^2)(\lambda^4 + \omega^2\lambda^2 + 27/4) = 0.$$

We may also compare our results with those of F. R. Moulton for the equilateral triangle configuration consisting of two finite bodies and an infinitesimal body. Choosing the unit of mass so that $m_1 + m_2 = 1$, and consequently $\omega^2 = 1$, Moulton[1] obtained the equation

$$\lambda^4 + \lambda^2 + 27/4\ m_1m_2 = 0.$$

Utilizing our choice of the unit of mass, this would assume the form

$$(39) \qquad \lambda^4 + \omega^2\lambda^2 + 27/4 = 0.$$

We observe that the functions appearing in the left members of the comparison equations (38) and (39) occur as factors of the form which we obtained in (35).

9. The characteristic equation of the x,y-system for the general isosceles trapezoid. If (26) is completely written out with A_{ij}, B_{ij}, C_{ij}, D_{ij} $(i,j = 1, 2, 3)$ replaced by their values as given by (24), the immediate possibility of obtaining some zero elements by easy combinations of certain rows or columns reveals itself. However the symmetric manner in which λ occurs will be disturbed and there will result a rather unwieldy mass of computation without the advantage of intermediate steps which

---

[1] F. R. Moulton, Celestial Mechanics, (2d revised ed.; New York: The MacMillan Company, 1931), p. 307.

may be readily checked. We shall therefore let the symmetry in λ remain undisturbed and make use of it to compute separately the coefficients of the various powers of λ. As each coefficient is obtained it may be compared with the results previously obtained for the particular cases of the square and the equilateral triangle.

The quantities, $+i\omega$ and $-i\omega$, occur among the roots of (27) for the general isosceles trapezoid. There are two zero roots of (26) for the special cases of the equilateral triangle and the square. Since o, o, $+i\omega$, and $-i\omega$ must appear among the characteristic exponents for the equations of variation, equation (26) must, also for the general isosceles trapezoid, have a double zero root. Therefore in expanding (26) we know that the coefficient of λ and the constant term must vanish.

Let

$$e = (e_1, e_2, e_3, e_4, e_5, e_6) = \begin{pmatrix} -\lambda^2 I + A & B \\ C & -\lambda^2 I + D \end{pmatrix},$$

$$f = (f_1, f_2, f_3, f_4, f_5, f_6) = \begin{pmatrix} 0 & 2\omega\lambda I \\ -2\omega\lambda I & 0 \end{pmatrix},$$

where e_i is the i-th column of the matrix e, and f_i is the i-th column of the matrix f. Then

$$(e + f) = (e_1 + f_1, e_2 + f_2, \cdots, e_6 + f_6) = \begin{pmatrix} -\lambda^2 I + A & 2\omega\lambda I + B \\ -2\omega\lambda I + C & -\lambda^2 I + D \end{pmatrix}.$$

The determinant of the matrix (e + f) may then be written

$$
\begin{aligned}
|e + f| &= |e_1 + f_1, e_2 + f_2, \cdots, e_6 + f_6| \\
&= |e_1 + f_1, \cdots, e_5 + f_5, \quad e_6 \quad| \\
&+ |e_1 + f_1, \cdots, e_5 + f_5, \quad f_6 \quad| \\
&= |e_1 + f_1, \cdots, e_4 + f_4, e_5, e_6| \\
&+ |e_1 + f_1, \cdots, e_4 + f_4, f_5, e_6| \\
&+ |e_1 + f_1, \cdots, e_4 + f_4, e_5, f_6| \\
&+ |e_1 + f_1, \cdots, e_4 + f_4, f_5, f_6|.
\end{aligned}
$$

We continue this process until we have a sum of sixty-four determinants each column of which contains either an element e_i or f_i $(i = 1, \cdots, 6)$.

These determinants may be grouped in the following manner:

	Columns from e	Columns from f	Number of Determinants
Type 0	6	0	1
Type 1	5	1	6
Type 2	4	2	15
Type 3	3	3	20
Type 4	2	4	15
Type 5	1	5	6
Type 6	0	6	1

The thirty-two determinants of Types 1, 3, 5 give only odd powers of λ. The thirty-two determinants of Types 0, 2, 4 and 6 give only even powers of λ. We shall first compute the coefficients of the odd powers of λ; later, those of the even powers of λ. In making the computation, we eliminate C_{ij} by means of the relation $C_{ij} = B_{ij}$ $(i,j = 1, 2, 3)$.

The determinant,

$$|e_1, e_2, \cdots, e_5, f_6|$$

which is one of the determinants of Type 1, may be put into the form

$$D = \begin{vmatrix} -\lambda^2 + A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & 0 \\ A_{21} & -\lambda^2 + A_{22} & A_{23} & B_{21} & B_{22} & 0 \\ A_{31} & A_{32} & -\lambda^2 + A_{33} & B_{31} & B_{32} & 2\omega\bar{\lambda} \\ B_{11} & B_{12} & B_{13} & -\lambda^2 + D_{11} & D_{12} & 0 \\ B_{21} & B_{22} & B_{23} & D_{21} & -\lambda^2 + D_{22} & 0 \\ B_{31} & B_{32} & B_{33} & D_{31} & D_{32} & -\lambda^2 + \bar{\lambda}^2 \end{vmatrix}.$$

This has been accomplished by replacing $2\omega\lambda$ by $2\omega\bar{\lambda}$ and the zero element in the sixth row, sixth column by $-\lambda^2 + \bar{\lambda}^2$, where $\bar{\lambda} = \lambda$.

This determinant expands into

$$(40) \qquad \lambda^{12} - a_1\lambda^{10} + a_2\lambda^8 - a_3\lambda^6 + a_4\lambda^4 - a_5\lambda^2 + a_6,$$

where a_i is equal to the sum of the i-rowed principal minors of the determinant formed by the omission of λ^2 from D.

Let us now replace $\bar{\lambda}$ by λ. We need consider only those terms in (40) containing the factor $2\omega\lambda$, since the others yield even powers of λ which must vanish in a determinant of Type 1.

We may therefore discard λ^{12} and $-a_1\lambda^{10}$. From $a_2\lambda^8$ we retain only $-2\omega\lambda^9 B_{33}$. From $-a_3\lambda^6$ we retain

$$2\omega\lambda^7 \left[\begin{vmatrix} A_{11} & A_{13} \\ B_{31} & B_{33} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ B_{32} & B_{33} \end{vmatrix} - \begin{vmatrix} B_{23} & D_{22} \\ B_{33} & D_{32} \end{vmatrix} - \begin{vmatrix} B_{13} & D_{11} \\ B_{33} & D_{31} \end{vmatrix} \right].$$

We proceed similarly with $a_4 \lambda^4$ and $- a_5 \lambda^2$. We need not consider a_6 since the terms that we retain would contribute to the coefficient of the first power of λ in (26), which vanishes.

Let us now consider the sum of the six determinants of Type 1. The coefficient of λ^9 in this sum is

$$2\omega(- B_{33} - B_{22} - B_{11} + B_{33} + B_{22} + B_{11}) = 0.$$

The coefficient of λ^7 vanishes identically in the A_{1j}, B_{1j}, and D_{1j}. The coefficients of λ^5 and λ^3 vanish when the A_{1j}, B_{1j}, and D_{1j} are replaced by their values as defined by (24).

The highest power of λ occurring in Type 3 is λ^7. The determinant,

$$|e_1, e_2, e_3, f_4, f_5, f_6|$$

which is one of the determinants of Type 3, may be put into the form,

$$- 8\omega^3\lambda^3 \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{vmatrix} .$$

The other nineteen may analogously be reduced to 3-rowed determinants. The coefficients of λ^7 and λ^5 in the sum of the determinants of Type 3 vanish identically in A_{1j}, B_{1j}, and D_{1j}. The coefficient of λ^3 from Type 3 determinants vanishes when the A_{1j}, B_{1j}, and D_{1j} are replaced by their values as defined by (24).

The sum of the determinants of Type 5 vanishes identically in the B_{1j}. Thus we have proven that there are no odd powers of λ in the characteristic equation (26).

By the observance of certain features of the elements of (26) a considerable amount of computation in obtaining the coefficients of even powers of λ is eliminated. It follows from the vanishing of the coefficients of the odd powers of λ and from the manner in which λ and ω enter into (26) that λ and ω appear in the terms of the expansion in the combination,

$$(-\lambda^2 + \omega^2)^h(\omega\lambda)^{2k} \qquad (0 \leqq h \leqq 6;\ 0 \leqq k \leqq 3).$$

Therefore if e_{hk} is the coefficient of $\lambda^{2h}(\omega\lambda)^{2k}$ and e'_{hk} is the coefficient of $\omega^{2h}(\omega\lambda)^{2k}$, we may write $e_{hk} = e'_{hk}$ for h even, and $e_{hk} = -e'_{hk}$ for h odd. Thus if we compute the coefficients of $\lambda^{10}\omega^2$, $\lambda^8\omega^4$, $\lambda^8\omega^2$, $\lambda^6\omega^4$, $\lambda^6\omega^2$, $\lambda^4\omega^2$, we may obtain at once the coefficients of $\lambda^2\omega^{10}$, $\lambda^4\omega^8$, $\lambda^2\omega^8$, $\lambda^4\omega^6$, $\lambda^2\omega^6$, $\lambda^2\omega^4$, respectively.

Further observation of the elements of (26) as defined in (24) shows that if 1 and 2 are replaced by each other in the subscripts of any one of the A_{1j}, B_{1j}, D_{1j}, there results an interchange of R_2 and R_5, T_2 and T_5, V_2 and $-V_5$. It is possible to group the terms entering into each coefficient in the expansion into the sum of two sets of terms in such a way that one set may be obtained from the other by an interchange of subscripts 1 and 2 in A_{1j}, B_{1j}, D_{1j}. All multiplications and sums may be completely obtained for one set, and the result for the other set found by a permutation of R_2 and R_5, T_2 and T_5, V_2 and $-V_5$.

The determinant of Type O may be expanded into the form

$$\lambda^{12} - a_1\lambda^{10} + a_2\lambda^8 - a_3\lambda^6 + a_4\lambda^4 - a_5\lambda^2 + a_6,$$

where a_i is equal to the sum of the i-rowed principal minors of the determinant

$$\begin{vmatrix} A & B \\ B & D \end{vmatrix}.$$

Since the constant term vanishes in (26), we may omit the computation of a_6.

The determinants of Type 2 are immediately reducible to the consideration of determinants of the fourth order, and those of Type 4 simplify at once to second order determinants. The determinant of Type 6 expands into $64\,\omega^6\lambda^6$.

After collecting the coefficients of like powers of λ from all types and using relation (5) and relation

$$R_1R_3 + R_2R_5 = R_0(R_1 + R_3), \tag{41}$$

which is derived from (5) and (6), to simplify the expressions, the characteristic equation has the form

$$\begin{aligned} &\lambda^{12} + \lambda^{10}(6\omega^2 + a_{50}) + \lambda^8(15\omega^4 + a_{41}\omega^2 + a_{40}) \\ &+ \lambda^6(20\omega^6 + a_{32}\omega^4 + a_{31}\omega^2 + a_{30}) + \lambda^4(15\omega^8 + a_{23}\omega^6 \\ &+ a_{22}\omega^4 + a_{21}\omega^2 + a_{20} + b_{20} + c_{20} + d_{20} + e_{20}) + \lambda^2(6\omega^{10} \\ &+ a_{14}\omega^8 + a_{13}\omega^6 + a_{12}\omega^4 + a_{11}\omega^2 + a_{10} + b_{10} + c_{10} \\ &+ d_{10} + e_{10}) = 0, \end{aligned} \tag{42}$$

where

$$a_{50} = -2MR_0 - 2m_1R_1 - 2m_3R_3,$$

$$a_{41} = -a_{14} = 3a_{50},$$

$$\begin{aligned} a_{40} = &-3M^2R_0^2 + 4MR_0(m_1R_1 + m_3R_3) - 8(m_1R_1 + m_3R_3)^2 \\ &+ 3(T_2 + T_5)[M^2R_0 + 2m_1m_3(R_1 + R_3)] - 12m_1m_3(R_2T_5 \\ &+ R_5T_2) - M^2/2\,(T_2^2 + T_5^2 + V_2^2 + V_5^2) - 4T_2T_5(m_1^2 \\ &+ m_3^2) + 4V_2V_5(m_1^2 - m_3^2), \end{aligned}$$

$$a_{32} = -a_{23} = 2a_{50},$$

$$a_{13} = a_{31} = 4a_{40} + 4MR_o(-MR_o - 4m_1R_1 - 4m_3R_3),$$

$$a_{22} = 6a_{40} + 8MR_o(-MR_o - 4m_1R_1 - 4m_3R_3),$$

$$\begin{aligned} a_{30} = {} & 2MR_o[2M^2R_o^2 + 3MR_o(m_1R_1 + m_3R_3) + 8(m_1R_1 + m_3R_3)^2] \\ & + 3M(T_2 + T_5)[-M^2R_o^2 - 2MR_o(m_1R_1 + m_3R_3) - 2m_1m_3R_1R_3] \\ & + 6m_1m_3M(R_2^2T_5 + R_5^2T_2) + M(T_2^2 + T_5^2 + V_2^2 + V_5^2) \\ & [M^2R_o/2 + m_1(2m_1 + m_3)R_1 + m_3(m_1 + 2m_3)R_3] \\ & + 2m_1m_3M[R_2(T_5^2 + V_5^2) + R_5(T_2^2 + V_2^2)] + 4T_2T_5[M(m_1^2 \\ & + m_3^2)R_o + m_1(2m_1^2 + m_1m_3 + m_3^2)R_1 + m_3(m_1^2 + m_1m_3 \\ & + 2m_3^2)R_3] + 2MV_2V_5[-M(m_1 - m_3)R_o - m_1(2m_1 - m_3)R_1 \\ & + m_3(2m_3 - m_1)R_3], \end{aligned}$$

$$a_{21} = -a_{12} = a_{30} + 8M^2R_o^2(m_1R_1 + m_3R_3),$$

$$a_{20} = 4M^2R_o^2[M^2R_o^2 - 2MR_o(m_1R_1 + m_3R_3) + 6(m_1R_1 + m_3R_3)^2],$$

$$\begin{aligned} b_{20} = {} & 6MR_o(T_2 + T_5)\{-M^3R_o^2 + M^2R_o(m_1R_1 + m_3R_3) - 4[m_1^2(2m_1 \\ & + m_3)R_1^2 + m_3^2(2m_3 + m_1)R_3^2 + 2m_1m_3MR_1R_3]\} + 12m_1m_3 \\ & MR_o[R_2T_5(MR_2 - 4m_1R_1 - 4m_3R_3) + R_5T_2(MR_5 - 4m_1R_1 \\ & - 4m_3R_3)], \end{aligned}$$

$$\begin{aligned} c_{20} = {} & (T_2^2 + T_5^2)[13M^4R_o^2/4 - M^3R_o(m_1R_1 + m_3R_3) + 8m_1^2(2m_1^2 \\ & + 2m_1m_3 + m_3^2)R_1^2 + 8m_3^2(2m_3^2 + 2m_1m_3 + m_1^2)R_3^2 \\ & + 8m_1m_3(7m_1^2 + 16m_1m_3 + 7m_3^2)R_1R_3] + 8m_1m_3R_2T_5^2 \\ & [4m_1^2R_1 + 4m_3^2R_3 - (5m_1^2 + 6m_1m_3 + 5m_3^2)R_2] \\ & + 8m_1m_3R_5T_2^2[4m_1^2R_1 + 4m_3^2R_3 - (5m_1^2 + 6m_1m_3 + 5m_3^2)R_5] \\ & + M(V_2^2 + V_5^2)[M^3R_o^2 - M^2R_o(m_1R_1 + m_3R_3) + 8m_1^3R_1^2 \\ & + 8m_3^3R_3^2 + 5m_1m_3MR_1R_3] + m_1m_3MR_2V_5^2[16(m_1R_1 + m_3R_3) \\ & - MR_2] + m_1m_3MR_5V_2^2[16(m_1R_1 + m_3R_3) - MR_5] + 2T_2T_5 \\ & \{13M^2R_o^2(m_1^2 + m_3^2) + 4MR_o[(-m_1^2 + 13\ m_1m_3 + 4m_3^2)m_1R_1 \end{aligned}$$

$$+ (-m_3^2 + 13m_1m_3 + 4m_1^2)m_3R_3] + 8m_1^2(2m_1^2 + 2m_1m_3 + m_3^2)R_1^2 + 8m_3^2(2m_3^2 + 2m_1m_3 + m_1^2)R_3^2 + 16m_1m_3(m_1^2 + m_1m_3 + m_3^2)R_1R_3\} + 2MV_2V_5\{4MR_o^2(-m_1^2 + m_3^2) + MR_o[m_1R_1(2m_1 - m_3) + m_3R_3(-2m_3 + m_1)] - 8m_1^3R_1^2 + 8m_3^3R_3^2 - 8m_1m_3(m_1 - m_3)R_1R_3\},$$

$$d_{20} = 3M^2/4\ [T_2(T_2^2 + V_2^2) + T_5(T_5^2 + V_5^2)][-M^2R_o - 2m_1m_3(R_1 + R_3)] + 3m_1m_3M^2[R_2T_5(T_5^2 + V_5^2) + R_5T_2(T_2^2 + V_2^2)] + 3(T_2T_5^2 + T_2^2T_5)[-3M^2R_o(m_1^2 + m_3^2) - 2m_1m_3R_1(7m_1^2 + 6m_1m_3 + 3m_3^2) - 2m_1m_3R_3(7m_3^2 + 6m_1m_3 + 3m_1^2)] + 12m_1m_3T_2T_5(m_1^2 - 2m_1m_3 + m_3^2)(R_2T_5 + R_5T_2) + 3(T_2V_5^2 + T_5V_2^2)[-M^4R_o/4 - m_1m_3M(5m_1 + m_3)R_1 - m_1m_3M(5m_3 + m_1)R_3] + 3m_1m_3M^2[R_2T_5V_2^2 + R_5T_2V_5^2] + 6V_2V_5(T_2 + T_5)(m_1^2 - m_3^2)[M^2R_o + 2m_1m_3(R_1 + R_3)] - 24m_1m_3V_2V_5(m_1^2 - m_3^2)(R_2T_5 + T_2R_5),$$

$$e_{20} = M^4/16\ (T_2^4 + T_5^4 + V_2^4 + V_5^4) + M^2(m_1^2 + m_3^2)(T_2T_5^3 + T_2^3T_5) + T_2^2T_5^2[4(m_1^2 + m_3^2)^2 + M^4/8] + M^4/8\ (T_2^2 + T_5^2)(V_2^2 + V_5^2) + M^2T_2T_5(m_1^2 + m_3^2)(V_2^2 + V_5^2) + M^2V_2V_5(-m_1^2 + m_3^2)(T_2^2 + T_5^2) + 8T_2T_5V_2V_5(-m_1^4 + m_3^4) + M^2V_2V_5(V_2^2 + V_5^2)(m_3^2 - m_1^2) + M^2V_2^2V_5^2/2\ (3m_1^2 - 2m_1m_3 + 3m_3^2),$$

$$a_{11} = 2(a_{20} + b_{20} + c_{20} + d_{20} + e_{20}) + 32M^2R_o^2[MR_o(m_1R_1 + m_3R_3) + (m_1R_1 + m_3R_3)^2] - 24M^3R_o^2(T_2 + T_5)(m_1R_1 + m_3R_3) + 4M^2(T_2^2 + T_5^2 + V_2^2 + V_5^2)[MR_o(m_1R_1 + m_3R_3) - m_1m_3R_1R_3] + 4m_1m_3M^2[R_2^2(T_5^2 + V_5^2) + R_5^2(T_2^2 + V_2^2)] + 16MR_oT_2T_5[m_1(2m_1^2 + m_1m_3 + m_3^2)R_1 + m_3(2m_3^2 + m_1m_3 + m_1^2)R_3] + 8M^2R_oV_2V_5[m_1(-2m_1 + m_3)R_1 + m_3(2m_3 - m_1)R_3],$$

$$a_{10} = 8M^3R_o^3(m_1R_1 + m_3R_3)[-MR_o - 4(m_1R_1 + m_3R_3)],$$

$$b_{10} = 12MR_o(T_2 + T_5)\{M^3R_o^2(m_1R_1 + m_3R_3) + 4R_o[m_1^2R_1^2(2m_1^2 + 4m_1m_3 + m_3^2) + m_3^2R_3^2(2m_3^2 + 4m_1m_3 + m_1^2) + 2m_1m_3R_1R_3(2m_1^2 + 3m_1m_3 + 2m_3^2)] - 4m_1m_3R_1R_3(m_1^2R_1 + m_3^2R_3)\} + 48m_1m_3MR_oR_2T_5[2m_1m_3R_o(R_1 + R_3) + R_2(m_1^2R_1 + m_3^2R_3)] + 48m_1m_3MR_oR_5T_2[2m_1m_3R_o(R_1 + R_3) + R_5(m_1^2R_1 + m_3^2R_3)],$$

$$c_{10} = M(T_2^2 + T_5^2)\{-13M^3R_o^2/2\ (m_1R_1 + m_3R_3) + 8R_o[-m_1^2R_1^2(2m_1^2 + 4m_1m_3 + m_3^2) - m_3^2R_3^2(2m_3^2 + 4m_1m_3 + m_1^2) - m_1m_3R_1R_3(3m_1^2 + 4m_1m_3 + 3m_3^2)] + 16m_1m_3R_1R_3(m_1^2R_1 + m_3^2R_3)\} + 8m_1m_3M(R_2^2T_5^2 + R_5^2T_2^2)[-2m_1^2R_1 - 2m_3^2R_3 - (m_1^2 + 6m_1m_3 + m_3^2)R_o] + 4MR_oT_2T_5\{MR_o[-m_1R_1(13m_1^2 + 38m_1m_3 + 11m_3^2) - m_3R_3(13m_3^2 + 38m_1m_3 + 11m_1^2)] + 4[-m_1^2R_1^2(2m_1^2 + 2m_1m_3 + m_3^2) - m_3^2R_3^2(2m_3^2 + 2m_1m_3 + m_1^2) - 2m_1m_3R_1R_3(m_1^2 + m_1m_3 + m_3^2)]\} + 2M(V_5^2 + V_2^2)\{-M^3R_o^2(m_1R_1 + m_3R_3) + 4R_o[-m_1^2R_1^2(2m_1^2 + 4m_1m_3 + m_3^2) - m_3^2R_3^2(2m_3^2 + 4m_1m_3 + m_1^2) - m_1m_3R_1R_3(3m_1^2 + 4m_1m_3 + 3m_3^2)] + 4m_1m_3R_1R_3[m_1R_1(2m_1 + m_3) + m_3R_3(2m_3 + m_1)]\} + 2m_1m_3MR_2V_5^2\{-R_o[M^2R_2 + 8m_1m_3(R_1 + R_3)] + 4R_2[-m_1(2m_1 + m_3)R_1 - m_3(m_1 + 2m_3)R_3]\} + 2m_1m_3MR_5V_2^2\{-R_o[M^2R_5 + 8m_1m_3(R_1 + R_3)] + 4R_5[-m_1(2m_1 + m_3)R_1 - m_3(m_1 + 2m_3)R_3]\} + 4M^2V_2V_5R_o\{MR_o[m_1(2m_1 - m_3)R_1 + m_3(-2m_3 + m_1)R_3] + 4(m_1R_1 + m_3R_3)(m_1^2R_1 - m_3^2R_3)\},$$

$$d_{10} = 3M^3/2\ (T_2^3 + T_5^3 + T_2V_2^2 + T_5V_5^2)[MR_o(m_1R_1 + m_3R_3) - m_1m_3R_1R_3] + 3/2\ m_1m_3M^3[R_2^2(T_5V_5^2 + T_5^3) + R_5^2(T_2V_2^2 + T_2^3)] + 3M(T_2T_5^2 + T_2^2T_5)\{M^2R_o[(3m_1 + 2m_3)m_1R_1 + (3m_3 + 2m_1)m_3R_3] - 2m_1m_3R_1R_3(m_1^2 + 6m_1m_3 + m_3^2)\}$$

$$+ 6m_1m_3M(m_1^2 + 6m_1m_3 + m_3^2)T_2T_5[R_2^2T_5 + R_5^2T_2]$$
$$+ 3M^3R_oV_2V_5(T_2 + T_5)[m_1R_1(-2m_1 + m_3) + m_3R_3(2m_3$$
$$- m_1)] + 3M^2[T_2V_5^2 + T_5V_2^2][2R_o(m_1^2 + 4m_1m_3 + m_3^2)$$
$$(m_1R_1 + m_3R_3) - m_1m_3MR_1R_3/2] + 3m_1m_3M^3/2\ [R_2^2T_2V_5^2$$
$$+ R_5^2T_5V_2^2],$$

$$e_{10} = M^3/4\ (T_2^2 + V_2^2)^2(-m_1^2R_1 - 2m_1m_3R_5 - m_3^2R_3) + M^3/4(T_5^2$$
$$+ V_5^2)^2(-m_1^2R_1 - 2m_1m_3R_2 - m_3^2R_3) + 2M(T_2^2V_5^2$$
$$+ T_5^2V_2^2)[-m_1m_3M^2R_o/2 + m_1^2(-m_1^2 - 2m_1m_3 + m_3^2)R_1$$
$$+ m_3^2(-m_3^2 - 2m_1m_3 + m_1^2)R_3] - 8m_1^2m_3^2M(R_2T_5^2V_2^2$$
$$+ R_5T_2^2V_5^2) + 2M^2V_2V_5(T_2^2 + V_2^2)[m_1^3R_1 - m_3^3R_3$$
$$+ m_1m_3(m_1 - m_3)R_5] + 2M^2V_2V_5(T_5^2 + V_5^2)[m_1^3R_1$$
$$- m_3^3R_3 + m_1m_3(m_1 - m_3)R_2] + M^2T_2T_5(T_2^2 + V_2^2)(-2m_1^3R_1$$
$$- 2m_3^3R_3 - m_1m_3MR_5) + M^2T_2T_5(T_5^2 + V_5^2)(-2m_1^3R_1$$
$$- 2m_3^3R_3 - m_1m_3MR_2) + 8MT_2T_5V_2V_5(m_1^4R_1 - m_3^4R_3)$$
$$+ MT_2^2T_5^2[4m_1m_3(-m_1^2 - 6m_1m_3 - m_3^2)R_o + m_1^2M(-3m_1$$
$$+ m_3)R_1 + m_3^2M(-3m_3 + m_1)R_3] + M^2V_2^2V_5^2[-m_1m_3MR_o$$
$$+ m_1^2(-3m_1 + m_3)R_1 + (-3m_3 + m_1)m_3^2R_3].$$

The expansion of (26) as given in (42) checks with our earlier results for the particular cases of the square and the equilateral triangle. If we replace the R_i, T_s, V_s, and m_r ($i = 0, 1, 2, 3, 5$; $s = 2, 5$; $r = 1, 3$) in equation (42) by their values for the equilaterial triangle as given by (34), then (42) reduces to the form given in (36). Similarly if we make use of the values of the square as given by (30), we obtain equation (32) from equation (42)

10. The characteristic exponents for the x,y-system. We shall now consider equation (42) for particular values of the parameter a, which has been defined earlier by means of (10) and (16).

The limiting values, a = 0 and a = 1, yield the equilateral triangle and square, respectively. For these we have already made exact computations. For the values .2, .4, .6, .8, we shall approximate numerical values of the coefficients of the various powers of λ in (42). To do this, we make use of our choice of units $m_1 = 1$, $r_1 = 1$ and employ relations (4), (7), (16), (18), (19), (20), (21), and (25).

A table of values of ω^2, R_1, T_s, V_s, m_r (i = 0, 1, 2, 3, 5; s = 2, 5; r = 1, 3) follows.

TABLE I

a	0	.2	.4	.6	.8	1
ω^2	2	2.03752	2.12081	2.23663	2.39079	2.70711
R_0	1	.99869	.98991	.96577	.91070	.67678
R_1	1	1	1	1	1	1
R_2	1	1.21552	1.30097	1.33516	1.30693	1
R_3	∞	36.93692	10.57834	4.95185	2.66887	1
R_5	1	.78187	.67884	.59637	.51447	.35355
T_2	2.25	3.13817	3.55822	3.79808	3.82954	3
T_5	2.25	1.50417	1.20337	.99129	.80971	.53033
V_2	1.29904	1.26308	1.10748	.88756	.59115	0
V_5	1.29904	1.12502	1.00130	.88931	.77077	.53033
m_1	1	1	1	1	1	1
m_3	0	.02009	.07122	.15796	.31261	1
m_3R_3	.75					

Through the use of this table we may put equation (42)

into the form

$$\lambda^{12} + d_5\lambda^{10} + d_4\lambda^8 + d_3\lambda^6 + d_2\lambda^4 + d_1\lambda^2 = 0, \tag{43}$$

where the d_1 are given by the following table.

TABLE II

a	d_5	d_4	d_3	d_2	d_1
0	4.5	26.25	72.88	152	182
.2	4.6657	27.25	77.00	160	192
.4	4.9765	29.72	86.94	186	229
.6	5.3821	33.36	102.25	229	296
.8	5.8945	38.51	125.35	299	411
1	6.8284	49.87	182.78	493	763

For both the square and the equilateral triangle equation (43) factors into the form

$$(\lambda^4 + \omega^2\lambda^2)(\lambda^4 + \omega^2\lambda^2 + h_o)(\lambda^4 + k_1\lambda^2 + k_o) = 0, \tag{44}$$

as equations (31) and (35) show. Let us make the assumption that (43) has factors of the form (44) for the general isosceles trapezoid. If so, then

$$\begin{aligned} d_5 &= 2\omega^2 + k_1, \\ d_4 &= \omega^4 + 2\omega^2 k_1 + h_o + k_o, \\ d_3 &= k_1\omega^4 + \omega^2(h_o + 2k_o) + h_o k_1, \\ d_2 &= k_o\omega^4 + \omega^2 h_o k_1 + h_o k_o, \\ d_1 &= \omega^2 h_o k_o. \end{aligned} \tag{45}$$

We use the first three equations of (45) to compute k_1, h_0, k_0. When these values are found for a = .2, .4, .6, .8 and substituted in the last two equations of (45), we obtain values for d_2 and d_1 which check with the ones appearing in Table II for the three significant figures there given.

We have shown for the particular cases considered that (43) factors into the form

$$(\lambda^4 + g\lambda^2)(\lambda^4 + h_1\lambda^2 + h_0)(\lambda^4 + k_1\lambda^2 + k_0) = 0,$$

where g and h_1 may be approximated by ω^2. For the square and the equilateral triangle, the equality $g = h_1 = \omega^2$ has been proven; for the other cases, our method of procedure constituted a proof that g and h_1 will at least lie near ω^2. By substituting $-\omega^2$ for λ^2 in (43) and by using closer approximations on the d_1, we verified that ω^2 and g coincide to five significant figures at least.

The factors of equation (43) for parameter values a = 0, .2, .4, .6, .8, 1 follow:

a = 0,

$$(\lambda^4 + 2\lambda^2)(\lambda^4 + 2\lambda^2 + 6.75)(\lambda^4 + .5\lambda^2 + 13.5) = 0;$$

a = .2,

$$(\lambda^4 + 2.0375\lambda^2)(\lambda^4 + 2.0375\lambda^2 + 6.76)(\lambda^4 + .5907\lambda^2 + 13.93) = 0;$$

a = .4,

$$(\lambda^4 + 2.1208\lambda^2)(\lambda^4 + 2.1208\lambda^2 + 7.30)(\lambda^4 + .7349\lambda^2 + 14.80) = 0; \quad (46)$$

a = .6,

$$(\lambda^4 + 2.2366\lambda^2)(\lambda^4 + 2.2366\lambda^2 + 8.25)(\lambda^4 + .9089\lambda^2 + 16.04) = 0;$$

a = .8,

$$(\lambda^4 + 2.3908\lambda^2)(\lambda^4 + 2.3908\lambda^2 + 9.67)(\lambda^4 + 1.1129\lambda^2 + 17.80) = 0;$$

a = 1,

$$(\lambda^4 + 2.7071\lambda^2)(\lambda^4 + 2.7071\lambda^2 + 12.73)(\lambda^4 + 1.4142\lambda^2 + 22.16) = 0.$$

The values of λ^2 for each equation in (46) are given by

$$\lambda^2 = 0,\ -g,\ c \pm di,\ \ c' \pm d'i,$$

where g is real and positive, and c, d, c', and d' are all real and different from zero. Consequently the values of λ have the form

$$\lambda = 0,\ 0,\ \pm i\sqrt{g}\ ,\ \ \pm\sqrt{c \pm di}\ \ ,\ \ \pm\sqrt{c' \pm d'i}\ . \tag{47}$$

11. Periodic solutions. We consider the z-system first. The transformation

$$\overline{z}_k = z_k\ , \qquad \overline{z}_{k+3} = dz_k/dt \qquad\qquad (k = 1, 2, 3),$$

reduces the system (23) to normal form. It is easily seen that the characteristic equation (27) may then be written

$$D(\lambda) = 0,$$

where $D(\lambda)$ is the determinant of the matrix

$$M_\lambda = \begin{pmatrix} -\lambda I & I \\ E & -\lambda I \end{pmatrix}, \tag{48}$$

in which I is the 3-rowed identity matrix and

$$E = (E_{ij}) \qquad\qquad (i,j = 1, 2, 3).$$

We seek to obtain the form of the solutions of (23) associated with the double roots which are given in (29). In order to do so we shall prove that all first minors of $D(i\omega)$ $D(-i\omega)$ vanish. Any matrix obtained from (48) by elementary transformations gives an equivalent matrix which has the same

rank[1] as (48).

The matrix

$$M'_{1\omega} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 + R_5 & 1 - R_2 & m_3(-R_2 + R_5) & \omega^2 & 0 & 0 \\ 1 - R_5 & -1 + R_2 & m_3(R_2 - R_5) & 0 & \omega^2 & 0 \\ R_5 - R_3 & R_2 - R_3 & 2m_3(R_0 - R_3) & 0 & 0 & \omega^2 \end{pmatrix}$$

is equivalent to $M_{1\omega}$ and to $M_{-i\omega}$. By a series of elementary transformations and through the use of (5) and (41) for simplification, we obtain the matrix

$$M''_{1\omega} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega^2 & \omega^2 & 0 \\ 0 & 0 & R_2 - 1 & 0 & \omega^2 & 0 \\ 0 & 0 & R_2 - R_3 & 0 & 0 & \omega^2 \end{pmatrix} .$$

One of the processes involved in transforming $M'_{1\omega}$ into $M''_{1\omega}$ was the multiplication of a column by $R_2 - 1$. The quantity $R_2 - 1$ vanishes for the equilateral triangle and for the square. With the exception of these two limiting cases, matrices $M_{1\omega}$ and $M_{-1\omega}$ are equivalent to $M''_{1\omega}$ and therefore have rank four.

For the square, the matrix

---

[1] A. Adrian Albert, Modern Higher Algebra (Chicago: The University of Chicago Press, 1937), p. 60.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega^2 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & \omega^2 \\ 0 & 0 & -1 + R_5 & 0 & 0 & \omega^2 \end{pmatrix}$$

is equivalent to $M'_{i\omega}$ and has rank four.

For the equilateral triangle, the matrix $M'_{i\omega}$ takes the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 1 - R_3 & 1 - R_3 & -3/2 & 0 & 0 & \omega^2 \end{pmatrix}.$$

The determinant and all first minors of this matrix will vanish provided we show that $O(1 - R_3)$ vanishes. It follows from a property of the real number system that we may write the equality $O = O \cdot m_3$. Therefore we have

$$O(1 - R_3) = O \cdot m_3(1 - R_3) = O(-3/2) = O.$$

The general solution of the system (23) is given by[1]

$$\bar{z}_k = (C_1 \eta_{k1} + C_2 \eta_{k2}) e^{i\omega t} + (C_3 \eta_{k3} + C_4 \eta_{k4}) e^{-i\omega t} + C_5 \delta_{k5} e^{i\nu t} + C_6 \delta_{k6} e^{-i\nu t} \qquad (k = 1, \cdots, 6), \tag{49}$$

---

[1] F. R. Moulton, Differential Equations (New York: The MacMillan Company, 1930), p. 248 and p. 254.

where C_j $(j = 1, 2, \cdots, 6)$ are arbitrary constants,

$$\nu = \sqrt{2m_1R_1 + 2m_3R_3} ,$$

η_{k1} and η_{k2} are second minors of $D(i\omega)$, η_{k3} and η_{k4} are second minors of $D(-i\omega)$, and δ_{k5} and δ_{k6} are first minors of $D(\nu)$ and $D(-\nu)$ respectively. The solution

$$\bar{z}_k = (C_1\eta_{k1} + C_2\eta_{k2})e^{i\omega t} + (C_3\eta_{k3} + C_4\eta_{k4})e^{-i\omega t}$$

is periodic of period $2\pi/\omega$. The solution

$$\bar{z}_k = C_5\delta_{k5}e^{i\nu t} + C_6\delta_{k6}e^{-i\nu t}$$

is periodic of period $2\pi/\nu$. If the periods $2\pi/\omega$ and $2\pi/\nu$ are commensurable, then the general solution (49) of the z-system is periodic.

Let us now consider the x,y-system for the particular parameter values, a = 0, .2, .4, .6, .8, 1. The eight roots

$$\pm\sqrt{c \pm di} , \qquad \pm\sqrt{c' \pm d'i} ,$$

are not pure imaginary. Therefore the general solution of the system (22) will not be periodic.

Following the method used in the case of the z-system, we may reduce the x,y-system to normal form by the transformation,

$$\bar{x}_k = x_k , \qquad \bar{x}_{k+3} = y_k ,$$

$$\bar{x}_{k+6} = dx_k/dt, \qquad \bar{x}_{k+9} = dy_k/dt \qquad (k = 1, 2, 3),$$

and express the characteristic equation (26) as $\bar{D}(\lambda) = 0$, where $\bar{D}(\lambda)$ is a 12-rowed determinant. A periodic solution will then be given by

$$\bar{x}_k = K_1 \delta_{k1} e^{i\sqrt{g}t} + K_2 \delta_{k2} e^{-i\sqrt{g}t} \qquad (k = 1,2,\cdots,12). \tag{50}$$

where K_1 and K_2 are arbitrary constants and δ_{k1} and δ_{k2} are first minors of $\bar{D}(i\sqrt{g})$ and $\bar{D}(-i\sqrt{g})$ respectively. If all first minors of $\bar{D}(0)$ vanish, there is associated with the double zero root a constant solution of the form

$$\bar{x}_k = K_3 \eta_{k3} + K_4 \eta_{k4} \qquad (k = 1,2, \cdots, 12), \tag{51}$$

where K_3 and K_4 are arbitrary, and η_{k3} and η_{k4} are second minors of $\bar{D}(0)$. If there exists at least one first minor of $\bar{D}(0)$ which is not zero, the double zero root gives rise to a constant solution

$$\bar{x}_k = K_3 \delta_{k3} \qquad (k = 1,2, \cdots, 12), \tag{52}$$

and to another solution which is a linear function[1] of t having no periodic properties. The solution consisting of the sum of solutions (50) and (51) or of (50) and (52), in the respective cases, will also be periodic.

The discussion of the periodic solutions of the x,y-system has been made for the parameter values, $a = 0, .2, .4, .6, .8, 1$. A substitution of any other value for a, $0 \geq a \geq 1$, in the expressions giving the coefficients of the various powers of λ in (48) yields an equation with numerical coefficients. The roots of this equation may then be approximated for this particular parameter value. However an observation of equations (45) and (46) shows that the coefficients are positive and increase as a takes on successively the values $0, .2, .4, .6, .8, 1$. Furthermore the relative increases are such that the discriminants of the last two quadratics in λ^2 in each of the equations in (46)

[1] Ibid., pp. 855-857.

remain negative. These facts seem to indicate that, in addition to the double zero root, there will be two pure imaginary roots which give periodic solutions and eight roots which do not give periodic solutions for all isosceles trapezoid configurations.

CHAPTER II

STABILITY OF THE EQUILATERAL TRIANGLE WITH CENTRAL BODY

12. Properties. We shall employ also in the second part of our paper the simplified notations for the distances r_{ij} $(i,j = 1, 2, 3, 4;\ i \neq j)$ between the masses; namely,

$$r_1 = r_{12},\ r_2 = r_{23},\ r_3 = r_{34},\ r_4 = r_{14},\ r_5 = r_{13},\ r_6 = r_{24}.$$

We assume

$$r_1 = r_2 = r_5, \qquad r_3 = r_4 = r_6. \tag{53}$$

MacMillan and Bartky showed that a quadrilateral satisfying (53) is an admissible one and that it consists of three equal masses at the vertices of an equilateral triangle and a fourth arbitrary mass at the center of gravity of the other three.[1]

We number the bodies so that m_1, m_2, and m_3 lie in a counter-clockwise order at the vertices of the triangle and m_4 at the center. We then have

$$m_1 = m_2 = m_3. \tag{54}$$

Let the distance r_o be defined by the relation

$$\omega^2 = \frac{M}{r_o^3}, \tag{55}$$

where $M = 3m_1 + m_4$. It follows from (53) and from the geometrical properties of an equilateral triangle that

---

[1] MacMillan and Bartky, op.cit., p. 872

(56) $$r_3 = r_4 = r_6 = \frac{r_1}{\sqrt{3}} .$$

The coordinates of the bodies are given by the constants

$$\xi_i^{(0)}, \qquad \eta_i^{(0)}, \qquad \zeta_i^{(0)},$$

where

$$\xi_1^{(0)} = -\frac{r_1}{2}, \quad \xi_2^{(0)} = \frac{r_1}{2}, \quad \xi_3^{(0)} = 0, \quad \xi_4^{(0)} = 0, \quad \eta_1^{(0)} = -\frac{r_1}{2\sqrt{3}},$$

$$\eta_2^{(0)} = -\frac{r_1}{2\sqrt{3}}, \quad \eta_3^{(0)} = \frac{r_1}{\sqrt{3}}, \quad \eta_4^{(0)} = 0, \quad \zeta_i^{(0)} = 0 \ (i = 1,2,3,4).$$

The triangle thus lies in the ξ, η-plane and the side r_1 is parallel to the ξ axis. The values

$$\xi_i = \xi_i^{(0)}, \qquad \eta_i = \eta_i^{(0)}, \qquad \zeta_i = \zeta_i^{(0)}$$

satisfy the differential equations (1) with the value of ω^2 given by

(57) $$\omega^2 = \frac{3m_1 + 3\sqrt{3}\, m_4}{r_1^3} .$$

We derive the value of r_o from the relation

(58) $$r_o^3 = \frac{(3m_1 + m_4)r_1^3}{3m_1 + 3\sqrt{3}\, m_4} ,$$

which is obtained from (55) and (57). The value of r_o as determined by (58) satisfies the following inequality relation established by MacMillan and Bartky[1] for the configuration under consideration,

$$r_4 \leq r_o \leq r_1 .$$

We choose the unit of distance so that $r_1 = 1$. We then have

---

[1] Ibid., p. 858.

$$r_1 = r_2 = r_5 = 1, \qquad r_3 = r_4 = r_6 = 1/\sqrt{3} . \tag{59}$$

The relation (57) then takes the form

$$\omega^2 = 3m_1 + 3\sqrt{3}\, m_4 . \tag{60}$$

13. The equations of variation. If m_2 and m_3 are eliminated by the use of (54) and the variables ξ_4, η_4, ζ_4, are eliminated by means of the center of gravity integrals, the equations of variation for the equilateral triangle configuration with one body at the center are

$$\begin{cases} d^2x_1/dt^2 - 2\omega dy_1/dt = A_{1j}x_j + B_{1j}y_j, \\ d^2y_1/dt^2 + 2\omega dx_1/dt = C_{1j}x_j + D_{1j}y_j, \end{cases} \tag{61}$$

$$d^2z_1/dt^2 = E_{1j}z_j \qquad (i,j = 1, 2, 3), \tag{62}$$

where the repeated subscripts indicate summation and where

$$\begin{aligned}
A_{11} &= A_{22} = \omega^2 + (7 + 15\sqrt{3})m_1/4 + (15\sqrt{3})m_4/4 , \\
A_{12} &= A_{21} = (-8 + 15\sqrt{3})m_1/4 , \\
A_{13} &= A_{23} = (1 + 15\sqrt{3})m_1/4 , \\
A_{31} &= A_{32} = (1 - 12\sqrt{3})m_1/4 , \\
A_{33} &= \omega^2 + (-2 - 12\sqrt{3})m_1/4 - (12\sqrt{3})m_4/4 , \\
\\
B_{11} &= -B_{22} = (3\sqrt{3} + 27)m_1/4 + 27m_4/4 , \\
B_{12} &= -B_{21} = 27m_1/4 , \\
B_{13} &= -B_{23} = (27 - 3\sqrt{3})m_1/4 , \\
B_{31} &= -B_{32} = (-3\sqrt{3})m_1/4 , \\
B_{33} &= 0, \\
\\
C_{ij} &= B_{ij} \qquad (i,j = 1, 2, 3),
\end{aligned} \tag{63}$$

$$D_{11} = D_{22} = \omega^2 + (1 - 3\sqrt{3})m_1/4 - (3\sqrt{3})m_4/4 ,$$
$$D_{12} = D_{21} = (4 - 3\sqrt{3})m_1/4 ,$$
$$D_{13} = D_{23} = (-5 - 3\sqrt{3})m_1/4 ,$$
$$D_{31} = D_{32} = (-5 + 24\sqrt{3})m_1/4 ,$$
$$D_{33} = \omega^2 + (10 + 24\sqrt{3})m_1/4 + (24\sqrt{3})m_4/4 ,$$

$$E_{11} = E_{22} = E_{33} = -(2 + 3\sqrt{3})m_1 - 3\sqrt{3}\, m_4 ,$$
$$E_{ij} = (1 - 3\sqrt{3})m_1 \qquad (i,j = 1,2,3;\ i \neq j).$$

14. The characteristic equation of the z-system. The characteristic equation of the z-system (62) is

$$(64) \qquad \begin{vmatrix} -\lambda^2 - (2 + 3\sqrt{3})m_1 - 3\sqrt{3}\, m_4 & (1 - 3\sqrt{3})m_1 & (1 - 3\sqrt{3})m_1 \\ (1 - 3\sqrt{3})m_1 & -\lambda^2 - (2 + 3\sqrt{3})m_1 - 3\sqrt{3}\, m_4 & (1 - 3\sqrt{3})m_1 \\ (1 - 3\sqrt{3})m_1 & (1 - 3\sqrt{3})m_1 & -\lambda^2 - (2 + 3\sqrt{3})m_1 - 3\sqrt{3}\, m_4 \end{vmatrix} = 0.$$

Expanding and simplifying by the use of (60), this equation reduces to

$$(\lambda^2 + \omega^2)^2(\lambda^2 + 3\sqrt{3}\, M) = 0.$$

The quantities ω^2 and $3\sqrt{3}\, M$ are distinct except when $m_1 = 0$. The roots of (64) are

$$(65) \qquad \lambda = \pm i\omega, \pm i\omega, \pm i\sqrt{3\sqrt{3}\, M} ,$$

which are all pure imaginary.

15. The characteristic equation of the x,y-system. The characteristic equation for the system (61) is

$$(66) \qquad \begin{vmatrix} -\lambda^2 I + A & 2\omega\lambda I + B \\ -2\omega\lambda I + C & -\lambda^2 I + D \end{vmatrix} = 0,$$

where I is the 3-rowed identity matrix and

$$A = (A_{ij}), \quad B = (B_{ij}), \quad C = (C_{ij}), \quad D = (D_{ij}) \qquad (i,j = 1,2,3).$$

Equation (66) may be put into the form

$$\begin{vmatrix} -\lambda^2 I + A' & 2\omega\lambda I + B' \\ -2\omega\lambda I + B' & -\lambda^2 I + D' \end{vmatrix} = 0,$$

where

$$A' = \begin{pmatrix} 9\omega^2/4 & 0 & 0 \\ 0 & \omega^2 + (15\sqrt{3})M/4 & (1 + 15\sqrt{3})m_1/4 \\ 0 & (-27\sqrt{3})M/4 & (3\omega^2 - 9\sqrt{3}\,M)/4 \end{pmatrix},$$

$$B' = \begin{pmatrix} 0 & 27M/4 & (27 - 3\sqrt{3})m_1/4 \\ (3\sqrt{3}m_1 + 27m_4)/4 & 0 & 0 \\ (-3\sqrt{3}\,\omega^2)/4 & 0 & 0 \end{pmatrix},$$

$$D' = \begin{pmatrix} (3\omega^2)/4 & 0 & 0 \\ 0 & \omega^2 - (3\sqrt{3})M/4 & (-5 - 3\sqrt{3})m_1/4 \\ 0 & (27\sqrt{3})M/4 & (9\omega^2 + 9\sqrt{3}\,M)/4 \end{pmatrix}.$$

The completed expansion is given by

$$\begin{aligned} &\lambda^{12} + 3\lambda^{10}[(4 - 3\sqrt{3})m_1 + 3\sqrt{3}\,m_4] + 9\lambda^8/2\,[6(5 - \sqrt{3})m_1^2 \\ &+ (-90 + 31\sqrt{3})m_1m_4 + 18m_4^2] + 27\lambda^6/4\,[(97 \\ &- 15\sqrt{3})m_1^3 + (-459 + 421\sqrt{3})m_1^2m_4 + 4(57 - 63\sqrt{3})m_1m_4^2 \\ &+ 12\sqrt{3}m_4^3] + 81/16\,m_1\lambda^4[(475 + 48\sqrt{3})m_1^3 + 2(-639 \\ &+ 364\sqrt{3})m_1^2m_4 + 3(2641 - 1104\sqrt{3})m_1m_4^2 + 72(-18 \\ &+ 5\sqrt{3})m_4^3] + 729/16\,\omega^2m_1^2\lambda^2[(31 + 12\sqrt{3})m_1^2 + 2(9 \\ &- 44\sqrt{3})m_1m_4 + 3(133 - 60\sqrt{3})m_4^2] = 0. \end{aligned} \tag{67}$$

The factor λ^2 occurs in this equation as it necessarily must, for there exist two zero roots of the characteristic equation by the same argument that was used in the isosceles trapezoid solution. The division of (67) by $\lambda^2 + \omega^2$ shows that $\lambda^2 + \omega^2$ enters as a factor also. Equation (67) may then be written in the form

$$\lambda^2(\lambda^2 + \omega^2)(\lambda^8 + A_1\lambda^6 + A_2\lambda^4 + A_3\lambda^2 + A_4) = 0, \tag{68}$$

where

$$\begin{aligned}
A_1 &= 3[3(1 - \sqrt{3})m_1 + 2\sqrt{3}\, m_4] , \\
A_2 &= 27/2\ [8m_1^2 + (-24 + 7\sqrt{3})m_1m_4 + 2m_4^2] , \\
A_3 &= 27m_1/4\ [(49 - 15\sqrt{3})m_1^2 + (-315 + 331\sqrt{3})m_1m_4 \\
&\quad + 18(5 - 6\sqrt{3})m_4^2] , \\
A_4 &= 729/16\ m_1^2[(31 + 12\sqrt{3})m_1^2 + 2(9 - 44\sqrt{3})m_1m_4 \\
&\quad + 3(133 - 60\sqrt{3})m_4^2] .
\end{aligned} \tag{69}$$

16. Limiting case of three bodies. Let us assume $m_4 = 0$. It then follows from (60) that $\omega^2 = 3m_1$ and equation (68) becomes

$$\begin{aligned}
&\lambda^2(\lambda^2 + \omega^2)[\lambda^8 + 3\omega^2\lambda^6(1 - \sqrt{3}) + 12\omega^4\lambda^4 \\
&\quad + \omega^6\lambda^2(49 - 15\sqrt{3})/4 + 9\omega^8/4\ (31/4 + 3\sqrt{3})] = 0.
\end{aligned} \tag{70}$$

The equation obtained by Buchanan for the equilateral triangle configuration of three bodies, which was given in Section 8, takes the following form when the three masses are equal,

$$\lambda^2(\lambda^2 + \omega^2)(\lambda^4 + \omega^2\lambda^2 + 9\omega^4/4) = 0. \tag{71}$$

In order to compare our result with (71), equation (70) may be written

$$(72)\qquad \lambda^2(\lambda^2+\omega^2)(\lambda^4+\omega^2\lambda^2+9\omega^4/4)[\lambda^4 + \omega^2\lambda^2(2-3\sqrt{3})+\omega^4(31/4+3\sqrt{3})]=0.$$

Thus the functions appearing in the left member of the comparison equation (71) occur as factors in the equation (72) which we have found.

If we write $\omega^2 = g$, the roots of (72) are of the same form as (47). The discussion of periodic solutions of the x,y-system which appears in Section 11 therefore applies here also. Upon setting up the matrix M_λ as defined by (48) for the case under consideration, we find that it has rank four for $\lambda = \pm i\omega$. It follows then, by the same development that was given in Section 11, that the general solution of the z-system is a sum of periodic solutions.

17. The characteristic expenents for the x,y-system. The case $m_1 = 0$ reduces the configuration to a single body and the case $m_4 = 0$ has just been discussed. We shall therefore assume for the remainder of our paper that $m_1 \neq 0$ and $m_4 \neq 0$, and select the unit of mass so that $m_4 = 1$.

Let us consider the last factor in equation (68) for which we shall use the notation

$$(73)\qquad f(\lambda) = \lambda^8 + A_1\lambda^6 + A_2\lambda^4 + A_3\lambda^2 + A_4 .$$

We seek to determine conditions under which there exist pure imaginary roots of $f(\lambda) = 0$. If we write $\lambda^2 = \sigma$, then $f(\lambda)$ becomes

$$(74)\qquad g(\sigma) = \sigma^4 + A_1\sigma^3 + A_2\sigma^2 + A_3\sigma + A_4 .$$

Negative roots of $g(\sigma) = 0$ will yield pairs of pure imaginary roots of $f(\lambda) = 0$. Since we have taken $m_4 = 1$, the coefficients

A_1, A_2, A_3, and A_4 are now functions of m_1. We examine each coefficient to determine the signs and the vanishing points as m_1 varies over all positive values. The results, correct to five decimal places, are given in Table III in terms of the coefficients of σ^k ($k = 0,1,\cdots, 4$) for the function $g(-\sigma)$.

TABLE III

THE FUNCTION $g(-\sigma)$.

Values of m_1	Signs and Vanishing Points of the Coefficients of				
	σ^4	σ^3	σ^2	σ	σ^0
$m_1 < .19368$	+	-	+	+	+
$m_1 = .19368$	+	-	0	+	+
$.19368 < m_1 < .36395$	+	-	-	+	+
$m_1 = .36395$	+	-	-	0	+
$.36395 < m_1 < 1.29077$	+	-	-	-	+
$m_1 = 1.29077$	+	-	0	-	+
$1.29077 < m_1 < 1.29788$	+	-	+	-	+
$m_1 = 1.29788$	+	-	+	-	0
$1.29788 < m_1 < 1.57735$	+	-	+	-	+
$m_1 = 1.57735$	+	0	+	-	+
$m_1 > 1.57735$	+	+	+	-	+

We apply Descartes' rule of signs. For m_1 on the interval $1.29077 < m_1 < 1.57735$, the number of negative real roots of $g(\sigma) = 0$ is four, two, or zero. For any other value of m_1, the number of negative real roots of $g(\sigma) = 0$ cannot exceed two.

For further investigation into the actual existence of

real roots, we write $g(\sigma)$ with the radical expressions in the coefficients replaced by decimal approximations,

$$
\begin{aligned}
g(\sigma) = {} & \sigma^4 + \sigma^3(-6.58846m_1 + 10.39231) + \sigma^2(108m_1^2 \\
& - 160.32120m_1 + 27) + m_1\sigma(155.3799m_1^2 \\
& + 1743.5845m_1 - 655.1651) + m_1^2(2359.436m_1^2 \\
& - 6124.533m_1 + 3974.456).
\end{aligned} \tag{75}
$$

The points at which the second derivative

$$g''(\sigma) = 12\sigma^2 + 6A_1\sigma + 2A_2$$

vanishes give the points of inflection of $g(\sigma)$. These points are

$$\sigma = -A_1/4 \pm \sqrt{(A_1/4)^2 - A_2/6}\ . \tag{76}$$

From (69) we find

$$(A_1/4)^2 - A_2/6 = 9/8\ [(2 - 9\sqrt{3})m_1^2 + (30 - 8\sqrt{3})m_1 + 2].$$

Now $(A_1/4)^2 - A_2/6$ takes on the value zero for one positive value m_1. This value computed correct to five decimal places is $m_1' = 1.30116$. An investigation of the values assumed by $(A_1/4)^2 - A_2/6$ for all positive values of m_1 shows that it is positive for $m_1 < m_1'$ and negative for $m_1 > m_1'$. Therefore there are two real points of inflection of $g(\sigma)$ for $m_1 < m_1'$, the two points of inflection coincide at $m_1 = m_1'$, and they are imaginary for $m_1 > m_1'$.

We shall examine in detail the interval $1.29077 < m_1 < 1.57735$. Let the derivative

$$g'(\sigma) = 4\sigma^3 + 3A_1\sigma^2 + 2A_2\sigma + A_3 \tag{77}$$

be designated by $g_1'(\sigma)$ for $m_1 = 1.29077$ and by $g_2'(\sigma)$ for

$m_1 = 1.57735$. We then have

$$g_1'(\sigma) = 4\sigma^3 + 5.6643\sigma^2 + 2393.46,$$

$$g_2'(\sigma) = 4\sigma^3 + 85.6500\sigma + 3914.46.$$

For m_1 defined by $1.29077 < m_1 \leqq 1.30116$, which is part of the interval that we are considering, the inflection points are real. An examination of (76) shows that $-A_1/4$ is an increasing function and $(A_1/4)^2 - A_2/6$ is a decreasing function of m_1 for $1.29077 \leqq m_1 \leqq 1.30116$. We find therefore from (76) that, for m_1 on this interval,

$$\sigma = -.4720 + .0171\,\epsilon_1 \pm (.4720 - .4720\,\epsilon_2),$$

where ϵ_1 and ϵ_2 increase from 0 to 1 as m_1 increases from 1.29077 to 1.30116. Thus for m_1 within the values $1.29077 \leqq m_1 \leqq 1.30116$, the two roots of $g''(\sigma) = 0$ are well within the interval $-1 < \sigma < 1$. For m_1 given by $1.30116 < m_1 < 1.57735$, there exist no real points of inflection.

Since A_3 is an increasing function of m_1 on the interval $1.29077 \leqq m_1 \leqq 1.57735$, the last term in $g'(\sigma)$ increases from 2393.46 to 3914.46 as m_1 increases from 1.29077 to 1.57735. Furthermore for m_1 so defined and σ on the interval $-1 < \sigma < 1$, it is easily seen, with the aid of the decimal values for A_1, A_2, and A_3 given in (75), that this last term, which is positive, determines the sign of $g'(\sigma)$. It follows therefore that, even when the points of inflection are real, the derivative does not change sign on $-1 < \sigma < 1$. As $g'(\sigma)$ changes from an increasing to a decreasing function and then from a decreasing to an increasing function at these real points of inflection, it remains positive.

Thus we have proven that for m_1 on the interval $1.29077 < m_1 < 1.57735$, $g'(\sigma)$ vanishes at only one real point and $g(\sigma)$ cuts the σ axis in at most two real places. This result combined with the previous conclusions drawn from Table III shows that there can never be more than two negative real roots of $g(\sigma) = 0$ for any positive m_1.

We shall now investigate whether or not two negative real roots of $g(\sigma) = 0$ exist for small positive values of m_1. For $m_1 = 0$, $g(\sigma)$ vanishes at a negative value of σ; namely, at $\sigma = -3\sqrt{3}$. Let us examine the function $g(\sigma)$ for $\sigma = -3\sqrt{3}$ and m_1 in the neighborhood of zero. We find from (74) with the use of the exact values of the coefficients given in (69) that

$$g(-3\sqrt{3}) = 81/16\, m_1^2[9(31 + 12\sqrt{3})m_1^2 + 2(171 - 494\sqrt{3})m_1 + 15(13 - 24\sqrt{3})]. \tag{78}$$

There are no terms in $g(-3\sqrt{3})$ which are independent of m_1 or of the first degree in m_1. For m_1 sufficiently small, we may neglect the terms which are of the third or fourth degrees in m_1 and consider only those of the second degree to determine the sign of $g(-3\sqrt{3})$. Therefore from $81/16\, m_1^2[15(13 - 24\sqrt{3})]$, we conclude that $g(-3\sqrt{3})$ is negative for m_1 sufficiently small. For any given value of m_1 and for σ sufficiently large and positive, $g(\sigma)$ is positive, for it is of even degree with the leading coefficient positive. Similarly for σ sufficiently large in magnitude and negative, $g(\sigma)$ is positive. Thus $g(\sigma)$ cuts the σ axis at two real points when m_1 is sufficiently small since we have found one value of σ for which it is negative. Furthermore, since this value of σ is negative and since $g(0)$ is positive, these two real points are negative real points.

We may determine from (78) the values of m_1 which will give $g(\sigma)$ a negative value at $\sigma = -3\sqrt{3}$. The right member of (78) has a positive factor $81/16\ m_1^2$ and a second factor which is given by

$$h(m_1) = 9(31 + 12\sqrt{3})m_1^2 + 2(171 - 494\sqrt{3})m_1 + 15(13 - 24\sqrt{3}).$$

The equation $h(m_1) = 0$ has a positive and a negative root. We are interested in the positive root only, of which the numerical approximation is $m_1'' = 3.22322$. For $m_1 < m_1''$, $h(m_1)$ is negative; and for $m_1 > m_1''$, it is positive. It follows therefore that for $m_1 < m_1''$, $g(\sigma)$ is negative at $\sigma = -3\sqrt{3}$. It is evident from the last column in Table III that $g(0)$ is positive except at $m_1 = 1.29788$ where it is zero. Therefore except at $m_1 = 1.29788$, $g(\sigma) = 0$ has two real negative roots for $m_1 < m_1''$ by the same argument that was used above. When $m_1 = 1.29788$, there is a real negative root of $g(\sigma) = 0$ and the other real root is equal to zero.

The value of m_1 for which we may demonstrate the existence of negative real roots may be extended to $m_1 = 3.6$. Let us consider the interval $3 \leqq m_1 \leqq 3.6$, which overlaps a part of the interval already found for the values of m_1 given by $3 \leqq m_1 < 3.22322$. The value of $g(\sigma)$ at $\sigma = -10$ is

$$g(-10) = k_1(m_1) + k_2(m_1),$$

where

$$k_1(m_1) = -2892.01m_1 + 2307.69$$

and

$$k_2(m_1) = m_1^2(2359.436m_1^2 - 7678.332m_1 - 2661.389).$$

Now $k_1(m_1)$ is a decreasing function of m_1 on the interval $3 \leqq m_1 \leqq 3.6$ and $k_2(m_1)$ is an increasing function on this interval. Since $k_1(3) = -6368.3$ and $k_2(3.6) = 3562.8$, the greatest possible value of $g(-10)$ for m_1 on the interval $3 \leqq m_1 \leqq 3.6$ is -2805.5. Thus we have found a negative value of σ which gives $g(\sigma)$ a negative value. Since $g(\sigma)$ is positive at $\sigma = 0$ and also for σ sufficiently large in magnitude and negative, there are two real negative roots of $g(\sigma) = 0$ for m_1 defined within the given limits.

We now show that for $m_1 = 3.7$, there are no real roots of $g(\sigma) = 0$. For this value of m_1 the points of inflection of $g(\sigma)$ are imaginary, and the derivative $g'(\sigma)$ vanishes at only one real point. Let $g(\sigma)$ be designated by $g_3(\sigma)$ for $m_1 = 3.7$. We then have

$$(79) \quad g_3(\sigma) = \sigma^4 - 13.9850\sigma^3 + 912.332\sigma^2 + 29316.02\sigma + 186380.7.$$

The numerical approximations of $g_3'(-10.8)$ and $g_3'(-10.7)$ are -322.8 and 88.5 respectively. The function $g_3(\sigma)$ therefore has its only critical point, a minimum point, in the interval between $\sigma = -10.8$ and $\sigma = -10.7$. As σ increases from -10.8 to -10.7, we find that

σ^4	decreases from	13604.9	to	13108.0 ,
$A_1\sigma^3$	decreases from	17617.1	to	17132.2 ,
$A_2\sigma^2$	decreases from	106414.4	to	104452.9 ,
$A_3\sigma$	increases from	-316613.0	to	-313681.4 ,

when $m_1 = 3.7$. The sum of the smallest values of the terms in the right member of (79) on this interval is 4460.8. Hence $g_3(\sigma)$ is positive at its minimum point. Therefore for $m_1 = 3.7$, $g(\sigma) = 0$ has no real roots and therefore no negative real roots.

Let the notations a, b, c, d be used for the roots of

$g(\sigma) = 0$. The zeros of $f(\lambda)$, defined by (73), are then given by

$$\lambda = \pm\sqrt{a}\ ,\ \pm\sqrt{b},\ \pm\sqrt{c}\ ,\ \pm\sqrt{d}\ . \tag{80}$$

We have shown that there are at most two of the quantities a, b, c, d which are real negative numbers; and consequently there are at most four pure imaginary roots of $f(\lambda) = 0$. Excluding the value of m_1 at which A_4 vanishes, there always exist four pure imaginary roots for $m_1 \leqq 3.6$. At the value of m_1 for which A_4 vanishes and which is approximated by $m_1 = 1.29788$, there are two pure imaginary roots. For the other values of m_1, the number of pure inaginary roots will be either four or zero since the points of inflection of $g(\sigma)$ are imaginary and since $g(\sigma)$ and $g'(\sigma)$ are both positive at $\sigma = 0$. The case for which none of the roots of $f(\lambda) = 0$ are pure imaginary exists since we have found a particular example, $m_1 = 3.7$.

The characteristic exponents of the x,y-system are

$$\lambda = 0,\ 0,\ \pm i\omega,\ \pm\sqrt{a},\ \pm\sqrt{b},\ \pm\sqrt{c},\ \pm\sqrt{d}\ , \tag{81}$$

which are obtained from (80) and from the roots of $\lambda^2(\lambda^2 + \omega^2) = 0$. The two pure imaginary exponents $\pm i\omega$ always exist. Thus the total number of pure imaginary exponents for the x,y-system is two, four, or six, depending upon the value of m_1.

18. Periodic solutions. Let the z-system (62) be put into normal form by the transformation

$$\bar{z}_k = z_k, \qquad \bar{z}_{k+3} = dz_k/dt \qquad (k = 1, 2, 3).$$

The characteristic equation (64) may then be written

$$D(\lambda) = 0\ ,$$

where $D(\lambda)$ is the determinant of the matrix M_λ defined by (48). From (65) it is seen that $+i\omega$ and $-i\omega$ are double roots of (64). As in Section 11, we investigate the ranks of the matrices $M_{i\omega}$ and $M_{-i\omega}$. By means of elementary transformations, we obtain the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega^2 & 0 & -\omega^2 \\ 0 & 0 & 0 & 0 & \omega^2 & -\omega^2 \\ 0 & 0 & 1-3\sqrt{3} & 0 & 0 & \omega^2 \end{pmatrix}$$

which is equivalent to $M_{i\omega}$ and to $M_{-i\omega}$ and which has rank four. The general solution of the system (62) is given by

$$\begin{aligned} \bar{z}_k = {} & (c_1\eta_{k1} + c_2\eta_{k2})e^{i\omega t} + (c_3\eta_{k3} + c_4\eta_{k4})e^{-i\omega t} \\ & + c_5\delta_{k5}e^{i\nu t} + c_6\delta_{k6}e^{-i\nu t} \qquad (k = 1,2\cdots,6), \end{aligned} \tag{82}$$

where $\nu = \sqrt{3\sqrt{3}\,M}$ and where we have employed the notations for the coefficients of the exponentials explained in Section 11. The solution

$$\bar{z}_k = (c_1\eta_{k1} + c_2\eta_{k2})e^{i\omega t} + (c_3\eta_{k3} + c_4\eta_{k4})e^{-i\omega t}$$

is periodic of period $2\pi/\omega$, and the solution

$$\bar{z}_k = c_5\delta_{k5}e^{i\nu t} + c_6\delta_{k6}e^{-i\nu t}$$

is periodic of period $2\pi/\nu$. If the periods $2\pi/\omega$ and $2\pi/\nu$ are commensurable, then the general solution (82) of the z-system is periodic.

We reduce the x,y-system, which is of order 12, to normal form by the transformation

$$\bar{x}_k = x_k , \qquad \bar{x}_{k+3} = y_k ,$$

$$\bar{x}_{k+6} = dx_k/dt , \qquad \bar{x}_{k+9} = dy_k/dt \qquad (k = 1,2,3).$$

The characteristic equation (66) may then be written as $\bar{D}(\lambda) = 0$, where $\bar{D}(\lambda)$ is a 12-rowed determinant. From the list of the characteristic exponents given in (81) it is seen that there is a double zero root of (66). If all first minors of $\bar{D}(0)$ vanish, there is associated with the double zero root a constant solution of the form

$$\bar{x}_k = K_1 \eta_{k1} + K_2 \eta_{k2} \qquad (k = 1,2,\cdots,12). \tag{83}$$

If all first minors of $\bar{D}(0)$ do not vanish, there is only one of the two solutions associated with the double zero root which is constant and which will be of the form

$$\bar{x}_k = K_1 \delta_{k1} \qquad (k = 1,2,\cdots,12). \tag{84}$$

The periodic solution

$$\bar{x}_k = K_3 \delta_{k3} e^{i\omega t} + K_4 \delta_{k4} e^{-i\omega t} \qquad (k = 1,2,\cdots,12), \tag{85}$$

associated with the roots $\pm i\omega$ always exists. On the basis of our earlier discussion we may assume for $m_1 = 1.29788$ that a is equal to zero and that b is negative real. For other values of m_1 none of the quantities a, b, c, d are zero and the number of negative real quantities among them is either two or zero. Let us assume that the two which are never negative or zero are c and d. Therefore there are no periodic solutions associated with the four characteristic exponents $\pm\sqrt{c}$, $\pm\sqrt{d}$. For values of m_1 for which

a and b are both negative there are the periodic solutions

$$\bar{x}_k = K_5 \delta_{k5} e^{\sqrt{a}\, t} + K_6 \delta_{k6} e^{-\sqrt{a}\, t} \tag{86}$$

and

$$\bar{x}_k = K_7 \delta_{k7} e^{\sqrt{b}\, t} + K_8 \delta_{k8} e^{-\sqrt{b}\, t} \qquad (k = 1,2,\cdots 12). \tag{87}$$

For $m_1 = 1.29788$, there is a periodic solution of the form given in (87)

Let us summarize this discussion on the periodic properties of the solutions of the x,y-system when the four bodies all exist. We have assumed that the unit of mass is so chosen that $m_4 = 1$ and have allowed m_1 to vary over all positive values. For $m_1 = 1.29788$, there are eight non-zero characteristic exponents, four of which yield periodic solutions. The form of the solutions given by the four zero roots depends upon the rank of $\bar{D}(0)$. For other values of m_1, there are ten non-zero characteristic exponents. For $m_1 \leq 3.6$ and different from 1.29788, six of these are associated with periodic solutions. For values of $m_1 > 3.6$, a detailed investigation has not been made beyond showing that there will be either two or six of the ten non-zero characteristic exponents giving periodic solutions. The sum of the constant solution given by (83) or (84) and any periodic solution is also periodic. Sums of periodic solutions whose periods are commensurable also yield periodic solutions.

SUMMARY

None of the configurations discussed in this paper are stable in the sense of a general displacement. However, we have shown that periodic orbits exist for certain types of displacements.

For both the isosceles trapezoid and the equilateral triangle with central body, the general solution of the z-system of the equations of variation is a linear combination of periodic solutions involving at most two distinct periods. One of these periods is $2\pi/\omega$ where ω is the angular velocity of the system. Therefore for small displacements in the z direction, the configurations are stable.

For the isosceles trapezoid, two characteristic exponents of the x,y-system are zero; so there is at least one constant solution. For parameter values a = 0, .2, .4, .6, .8, 1, the ten remaining characteristic exponents include two pure imaginary quantities $\pm i\sqrt{g}$ and eight which do not give periodic solutions. For these values of a there thus exist periodic solutions of period $2\pi/\sqrt{g}$ in the plane of the configuration. The manner in which the coefficients vary in the characteristic equations seems to indicate that these results hold for all isosceles trapezoid solutions. For a = 0 or 1, the existence of periodic solutions of period $2\pi/\omega$ in x,y,z-space follows from $g = \omega^2$. For a = .2, .4, .6, .8, we have found that the value of g coincides with the value of ω^2 to five significant figures at least; but since approximations have been used, we have not proven equality.

The characteristic roots of the x,y-system for the equilateral triangle with central body are here tabulated:

	Zero	Pure Imaginary	Other Types
$m_1 \leqq 3.6m_4$, $m_1 \neq 1.29788m_4$	2	6	4
$m_1 = 1.29788m_4$	4	4	4
$m_1 > 3.6m_4$	2	2 or 6	8 or 4

Thus there always exists at least one constant solution. It is to be observed that for certain values of the masses there are six pure imaginary roots yielding periodic solutions. The values $\pm i\omega$ always occur. Therefore periodic solutions of period $2\pi/\omega$ exist in the x,y-plane and consequently also in x,y,z-space.

VITA

Adrienne Sophie Rayl was born September 25, 1898, in New Orleans, Louisiana. She attended the public schools of New Orleans and graduated from the New Orleans Normal School in 1917.

In September, 1917, she was appointed as an elementary teacher in the public schools of New Orleans. She enrolled in the courses for teachers and in summer school classes at Tulane University and received the degree of Bachelor of Arts in Education in 1924. In September, 1929, she was transferred to the department of mathematics in the public high schools of New Orleans.

She had graduate work in mathematics in summer school and in afternoon classes at Tulane University under Professors Anderson, Buchanan, Duren, and Weiss, and received the degree of Master of Arts in 1934. During the summer quarters of 1934 and 1935, the summer and autumn quarters of 1936, the winter, spring, and summer quarters of 1937, and the summer quarter of 1938, she had graduate work in mathematics at the University of Chicago under Professors Albert, Barnard, Bartky, Bliss, Dickson, Graves, Lane, Logsdon, McShane, Reid, and Wilcox.

She wishes to express her gratitude to the faculties at Tulane University and at the University of Chicago for their inspiration in the field of mathematics. She thanks Professor Bartky particularly for his valuable suggestions and guidance in the preparation of this dissertation.